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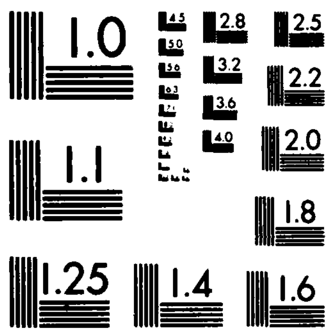

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FINAL REPORT ON SCIENTIFIC ACTIVITIES

PURSUANT TO THE PROVISIONS OF

AFOSR GRANT 79-0018-A

DURING THE PERIOD

NOVEMBER 1, 1982 TO OCTOBER 31, 1983

D. L. RUSSELL, PRINCIPAL INVESTIGATOR

DEPARTMENT OF MATHEMATICS

UNIVERSITY OF WISCONSIN

MADISON

1. General Remarks.

During the period November 1, 1982 to October 31, 1983, the Principal Investigator, in cooperation with several research assistants, carried out a program of mathematical research in the general area of control theory of partial differential equations and began the operation of the Modelling, Information Processing and Control Facility here at the University of Wisconsin, whose initial equipment acquisitions and continuing operation have been funded, in part, by AFOSR under Grant 79-0018. The program now involves three distinct phases, all of which are under some degree of development. There is the fundamental program of research on the control theory of distributed parameter systems and the related program of research on self-excited oscillations related to flutter phenomena, the specific research program aimed at the development and improvement of control and identification strategies in connection with wing flutter problems, and the new area of distributed parameter model development and calibration in connection with the MIPAC facility just described.

During the period just noted our work has resulted in two scientific papers which form the greater part of this report. The first of these, "The Dirichlet-Neumann Boundary Control Problem Associated with Maxwell's Equations in a Cylindrical Region" has been developed in connection with the first phase of our research program and was presented to the IEEE Conference on Decision and Control in December, 1983. The Principal Investigator is being assisted in further

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19. ABSTRACT (Continue on reverse if necessary and identify by block number)

Research during this period was carried out in the general area of control theory of partial differential equations and the operation of the Modelling, Information Processing and Control Facility at the University of Wisconsin was begun. The program involves three distinct phases, all of which are under some degree of development. There is the fundamental program of research on the control theory of distributed parameter systems and the related program of research on self-excited oscillations related to flutter phenomena, the specific research program aimed at the development and improvement of control and identification strategies in connection with wing flutter problems, and the new area of distributed parameter model development and calibration in connection with the MIPAC facility.

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development of this line of research by Research Assistant Katherine Kime.

The second paper, "Dual Paley-Wiener Spaces and "Regular" Nonharmonic Fourier Series" is an outgrowth of earlier work on the problem of closed loop eigenvalue specification in distributed parameter systems of hyperbolic type. Research Assistant Helen Baron, who has been partially supported under this grant, is continuing work in this area in connection with other classes of distributed parameter systems.

In addition to those already mentioned, Research Assistants Richard Rebarber and Robert Acar have received partial support under this grant and are continuing work in the areas of relative controllability of distributed parameter systems and coefficient identification in distributed parameter systems, respectively.

Computational studies performed with the UW MACC 1110 Computer and funded under this grant have enabled us to develop a new and effective procedure for identification of the period of an oscillatory disturbance, paving the way for adaptive control of certain flutter phenomena.

2. Technical Appendices.

The remainder of this report consists of two technical appendices as follows:

Appendix I: The Dirichlet-Neumann Boundary Control Problem Associated with Maxwell's Equations in a Cylindrical Region

Appendix II: Dual Paley-Wiener Spaces and "Regular" Nonharmonic Fourier Series

Both of these are authored by the Principal Investigator.

APPENDIX I

The Dirichlet-Neumann Boundary Control Problem Associated with Maxwell's Equations in a Cylindrical Region

This work was also supported in part by the Army Research Office under Contract DAAG29-80-C-0041.

SIGNIFICANCE AND EXPLANATION

This paper concerns the controllability of the Maxwell electromagnetic equations in a cylindrical spatial region by means of controlling currents caused to flow on the boundary of the region. Here controllability refers to the ability to transfer from electric and magnetic fields, given at the initial instant, to corresponding fields prescribed at a later instant.

Studies of this type are significant in relation to wave guides, EM-pulse devices, radar non-relective (stealth) aircraft, controlled thermonuclear fusion and many other important applications.

THE DIRICHLET-NEUMANN BOUNDARY CONTROL PROBLEM ASSOCIATED WITH MAXWELL'S EQUATIONS IN A CYLINDRICAL REGION

D. L. Russell

1. BACKGROUND.

In this paper we consider a region $\Omega \subseteq R^3$, not necessarily bounded, having piecewise smooth boundary Γ and almost everywhere uniquely defined unit exterior normal vector $\vec{\nu} = \vec{\nu}(x,y,z)$, $(x,y,z) \in \Gamma$. It is assumed that the region Ω is occupied by a medium having constant electrical permitivity ϵ and constant magnetic permeability μ. We have then, in Ω, the paired electric and magnetic fields

$$\vec{E} = \vec{E}(x,y,z,t) ,$$

$$\vec{H} = \vec{H}(x,y,z,t) ,$$

having finite <u>energy</u>

$$E(t) = \tfrac{1}{2} \iiint_{\Omega} (\epsilon \|\vec{E}\|^2 + \mu \|\vec{H}\|^2) dv , \tag{1.1}$$

where $\| \ \|$ denotes the usual Euclidean norm in R^3. As is well known ([4], [9]), $\vec{E}$ and $\vec{H}$ satisfy, in Ω, Maxwell's equations

$$\operatorname{curl} \vec{H} = \epsilon \frac{\partial \vec{E}}{\partial t} , \tag{1.2}$$

$$\operatorname{curl} \vec{E} = -\mu \frac{\partial \vec{H}}{\partial t} , \tag{1.3}$$

$$\operatorname{div} \vec{E} = \rho , \tag{1.4}$$

$$\operatorname{div} \vec{H} = 0 , \tag{1.5}$$

where $\rho = \rho(x,y,z,t)$ is the electrical charge density in Ω - which is zero throughout this paper. (That equation (1.5) might eventually have to be modified to account for magnetic monopoles will trouble us not at all here!)

Control problems associated with Maxwell's equations have been of interest primarily in connection with nuclear fusion applications - in which case ρ is not identically equal to zero and the Maxwell equations are coupled with the dynamical equations governing the plasma evolution. In this connection we cite the work of P. K. C. Wang [29], [30], [31]. The point of view which we take here is that we cannot hope to treat these more complicated problems until we have a firmer grasp on the control theory of Maxwell's system in its own right. In this direction some work on controllability with control influence distributed throughout Ω has been carried out by G. Chen [2], [3]. We are primarily concerned here with the possibility of influencing the evolution of the fields $\vec{E}$ and $\vec{H}$ by means of an externally determined current $\vec{J}(x,y,z,t)$ flowing tangentially in Γ so that

$$\vec{J}(x,y,z,t)\cdot\vec{\nu}(x,y,z) = 0 , \tag{1.6}$$

for $(x,y,z) \in \Gamma$ where $\vec{\nu}(x,y,z)$ is defined. We will assume that the normal component of $\vec{E}$ vanishes outside Ω and that no charge is permitted to accumulate on Γ. Then we have the boundary conditions (see e.g. [4], [28])

$$\epsilon\vec{E}(x,y,z,t)\cdot\vec{\nu}(x,y,z) = 0 \tag{1.7}$$

$$\mu\vec{H}_\tau(x,y,z,t) = \vec{\nu}(x,y,z) \times \vec{J}(x,y,z,t) \tag{1.8}$$

for $(x,y,z) \in \Gamma$ such that $\vec{\nu}(x,y,z)$ is well-defined. Here, and subsequently, the subscript τ refers to the component of the vector in question which is tangential to Γ. Similarly, the subscript ν will denote the normal component (thus (1.7) is the same as $\vec{E}_\nu = 0$). Writing

$$\vec{E} = \vec{E}_\nu + \vec{E}_\tau = \vec{E}_\tau \quad \text{on} \quad \Gamma ,$$
$$\vec{H} = \vec{H}_\nu + \vec{H}_\tau$$
$$\vec{J} = \vec{J}_\nu + \vec{J}_\tau = \vec{J}_\tau \quad \text{on} \quad \Gamma ,$$

we see that (1.8) becomes $\mu\vec{H}_\tau = \vec{\nu} \times \vec{J}_\tau$, so that $\vec{H}_\tau$ is a vector tangential to Γ and perpendicular to $\vec{J} = \vec{J}_\tau$.

The state space in which we study solutions of the above system will be denoted by $H_{E,d}(\Omega)$; it is a closed subspace of the space $H_E(\Omega)$ of square integrable six-dimensional fields $(\vec{E}(x,y,z,t), \vec{H}(x,y,z,t))$ with the inner product and norm

$$\langle(\vec{E}_1,\vec{H}_1);(\vec{E}_2,\vec{H}_2)\rangle \equiv \iiint_{\Omega} (\epsilon\vec{E}_1\cdot\vec{E}_2 + \mu\vec{H}_1\cdot\vec{H}_2)dv$$

$$\|(\vec{E},\vec{H})\|^2 = \langle(\vec{E},\vec{H});(\vec{E};\vec{H})\rangle \ . \qquad (1.9)$$

Clearly $H_{\mathbf{E}}(\Omega)$ is a real Hilbert space with this inner product. Where a complex space is required, we employ conjugation as usual. The state space $H_{\mathbf{E},d}(\Omega)$ is the closed span in $H_{\mathbf{E}}(\Omega)$ of those continuously differentiable fields $(\vec{E}(x,y,z,t), \vec{H}(x,y,z,t))$ for which

$$\operatorname{div} \vec{E} = \frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} + \frac{\partial E_z}{\partial z} = 0 \ ,$$

$$\operatorname{div} \vec{H} = \frac{\partial H_x}{\partial x} + \frac{\partial H_y}{\partial y} + \frac{\partial H_z}{\partial z} = 0 \ .$$

If $\vec{E}_0,\vec{H}_0$ and $\vec{E}_1,\vec{H}_1$ are two smooth solution pairs for (1.2)-(1.5), (1.7), (1.8), the first corresponding to $\vec{J} \equiv 0$ on Γ, we see easily that

$$\frac{d}{dt}\langle(\vec{E}_0,\vec{H}_0);(\vec{E}_1,\vec{H}_1)\rangle =$$

$$= \iiint_{\Omega} \left(\epsilon\left[\vec{E}_0\cdot \frac{\partial\vec{E}_1}{\partial t} + \frac{\partial\vec{E}_0}{\partial t}\cdot \vec{E}_1\right] + \mu\left[\vec{H}_0\cdot \frac{\partial\vec{H}_1}{\partial t} + \frac{\partial\vec{H}_0}{\partial t}\cdot \vec{H}_1\right]\right)dv$$

$$= \text{(using (1.2), (1.3))} =$$

$$= \iiint_{\Omega} (\vec{E}_0\cdot \operatorname{curl} \vec{H}_1 - \operatorname{curl} \vec{E}_0\cdot \vec{H}_1 + \operatorname{curl} \vec{H}_0\cdot \vec{E}_1 - \vec{H}_0\cdot \operatorname{curl} \vec{E}_1)dv$$

$$= \text{(using } \operatorname{div}(E \times H) = \operatorname{curl}\vec{E}\cdot\vec{H} - \vec{E}\cdot\operatorname{curl}\vec{H})$$

$$= -\iiint_{\Omega} [\operatorname{div}(\vec{E}_0 \times \vec{H}_1) + \operatorname{div}(\vec{E}_1 \times \vec{H}_0)]dv$$

$$= -\iint_{\Gamma} (\vec{E}_0 \times \vec{H}_1 + \vec{E}_1 \times \vec{H}_0)\cdot \nu ds = \text{(using (1.7))}$$

$$-\iint_{\Gamma} (\vec{E}_{0\tau} \times \vec{H}_{1\tau} + E_{0\tau} \times \vec{H}_{1\nu} + \vec{E}_{1\tau} \times \vec{H}_{0\tau} + \vec{E}_{1\tau} \times \vec{H}_{0\nu})\cdot \nu ds$$

$$= - \iint_\Gamma (\vec{E}_{0\tau} \times \vec{H}_{1\tau} + \vec{E}_{1\tau} \times \vec{H}_{0\tau}) \cdot \nu ds$$

= (using (1.8) and noting that $J \equiv 0$ for $\vec{E}_0, \vec{H}_0$)

$$= - \iint_\Gamma (\vec{E}_{0\tau} \cdot \vec{J}) ds \; . \qquad (1.10)$$

If we go through the same computation with $\vec{E}_0, \vec{H}_0, \vec{E}_1, \vec{H}_1$ both replaced by the same $\vec{E}, \vec{H}$ satisfying (1.2)-(1.5), (1.7), (1.8) we find that

$$\frac{d\mathbb{E}}{dt} = - \iint_\Gamma (\vec{E} \times \vec{H}) \cdot \vec{\nu} ds = - \iint_\Gamma \vec{E}_\tau \cdot \vec{J} ds \; . \qquad (1.11)$$

For $\vec{J} \equiv 0$ generalized solutions of (1.2)-(1.5), (1.7), (1.8) can be discussed in the general context of partial differential equations and strongly continuous semigroups. The generator

$$A(\vec{E},\vec{H}) = \left(\frac{1}{\varepsilon} \operatorname{curl} \vec{H}, - \frac{1}{\mu} \operatorname{curl} \vec{E}\right) \qquad (1.12)$$

with domain consisting of $\vec{E},\vec{H}$ in the Sobolev space $H^1_{\mathbb{E},d}(\Omega) (= H_{\mathbb{E},d}(\Omega) \cap H^1(\Omega))$ having zero divergence and satisfying (cf. (1.7), (1.8))

$$\vec{E}_\nu\big|_\Gamma = 0, \quad \vec{H}_\tau\big|_\Gamma = 0 \; , \qquad (1.13)$$

is antisymmetric and generates a group of isometries in $H_{\mathbb{E},d}(\Omega)$. (See [32], [33], [34] for related work.) Sufficient conditions on $\vec{J}$ so that solutions of the inhomogeneous system (1.2)-(1.5), (1.7), (1.8) lie in $H_{\mathbb{E},d}(\Omega)$ and are strongly continuous there may be obtained much as in [18], [19] but it is not easy to specify necessary and sufficient conditions. Indeed, this is already difficult for the much simpler, but related, wave equation

$$\mu\varepsilon \frac{\partial^2 w}{\partial t^2} = \frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} + \frac{\partial^2 w}{\partial z^2}$$

with boundary forcing terms. We will make some comments related to this in Section 6.

2. CONTROL PROBLEMS IN A CYLINDRICAL REGION

The main point in this paper is to study the question of controllability of the electromagnetic field $\vec{E},\vec{H}$ by means of the boundary current $\vec{J} = \vec{J}_\tau$. By controllability we mean the possibility of transferring an initial field $\vec{E}(x,y,z,0)$, $\vec{H}(x,y,z,0) \in H_{E,d}(\Omega)$, given at time $t = 0$, to a prescribed terminal field $\vec{E}(x,y,z,T)$, $\vec{H}(x,y,z,T) \in H_{E,d}(\Omega)$, specified at $t = T > 0$, by means of a suitable control current $\vec{J}(x,y,z,t)$ defined for $(x,y,z) \in \Gamma$, $t \in [0,T]$. Because the homogeneous Maxwell equations correspond to a group of isometries in $H_{E,d}(\Omega)$, it is enough to consider the special case wherein

$$\vec{E}(x,y,z,0) \equiv 0 \ , \tag{2.1}$$

$$\vec{H}(x,y,z,0) \equiv 0 \ . \tag{2.2}$$

For a given space, J, of admissible control currents $\vec{J}(x,y,z,t) = \vec{J}_\tau(x,y,z,t)$ defined on $\Gamma \times [0,T]$ we define the reachable set $R(T,J)$ to be the subspace of $H_{E,d}(\Omega)$ consisting of states reachable from the zero initial state using controls $\vec{J} \in J$. Following earlier definitions ([8], [26]), our system is <u>approximately controllable in time T</u> if $R(T,J)$ is dense in $H_{E,d}(\Omega)$ and <u>exactly controllable in time T</u> if $R(T,J) = H_{E,d}(\Omega)$ (or some precisely designated subspace of $H_{E,d}(\Omega)$).

At this writing we are not able to discuss the general three dimensional problem wherein the vector fields $\vec{E}$ and $\vec{H}$ are unrestricted, except as stipulated heretofore, and Ω has a general geometry. We hope in later work to consider at least some three dimensional cases which arise for special domains Ω. But for now we must content ourselves with the case in which Ω is a cylinder:

$$\Omega = R \times (-\infty,\infty) = \{(x,y,z)\,|\,(x,y) \in R \subset R^2,\ z \text{ real}\}$$

where R is an open connected region in R^2 with piecewise smooth boundary B. Thus

$$\partial\Omega = \partial R \times (-\infty,\infty) = B \times (-\infty,\infty) \ .$$

Even here we can give results only for special two dimensional regions R.

The two dimensional problem in the cylinder $\Omega = R \times (-\infty,\infty)$ occurs when we confine attention to fields

$$\vec{E} = \vec{E}(x,y,t),\ \vec{H} = \vec{H}(x,y,t)$$

which do not depend on the coordinate z corresponding to the axial, or longitudinal, direction of the cylinder. (Note that this is not at all the same thing as requiring that E_z, H_z, the field components in the z direction, should be zero.) We correspondingly consider only control currents

$$\vec{J} = \vec{J}(x,y,t)$$

which do not depend upon z.

Of course the energy E in Ω is infinite under the above circumstances if $\vec{E}$, $\vec{H}$ are not identically zero. We redefine E to be the energy per unit length of cylinder:

$$E(t) = \tfrac{1}{2} \iint_R (\varepsilon \|\vec{E}(x,y,t)\|^2 + \mu \|\vec{H}(x,y,t)\|^2) dxdy \; . \tag{2.3}$$

The space $H_{E,d}(\Omega)$ is now replaced by $H_{E,d}(R)$. Because

$$\frac{\partial E_z(x,y,t)}{\partial z} \equiv 0, \quad \frac{\partial H_z(x,y,t)}{\partial z} \equiv 0$$

we have

$$\operatorname{div} \vec{E} = \frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y}, \quad \operatorname{div} \vec{H} = \frac{\partial H_x}{\partial x} + \frac{\partial H_y}{\partial y} \; . \tag{2.4}$$

The curl expressions simplify to

$$\operatorname{curl} \vec{E} = \left(\frac{\partial E_z}{\partial y}, -\frac{\partial E_z}{\partial x}, \frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y}\right) ,$$

$$\operatorname{curl} \vec{H} = \left(\frac{\partial H_z}{\partial y}, -\frac{\partial H_z}{\partial x}, \frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y}\right) ,$$

so that the equations (1.2), (1.3) become

$$\begin{array}{llll} \text{(i)} & \varepsilon \dfrac{\partial E_x}{\partial t} = \dfrac{\partial H_z}{\partial y} & \text{(iv)} & \mu \dfrac{\partial H_x}{\partial t} = -\dfrac{\partial E_z}{\partial y} \\[2ex] \text{(ii)} & \varepsilon \dfrac{\partial E_y}{\partial t} = -\dfrac{\partial H_z}{\partial x} & \text{(v)} & \mu \dfrac{\partial H_y}{\partial t} = \dfrac{\partial E_z}{\partial x} \\[2ex] \text{(iii)} & \varepsilon \dfrac{\partial E_z}{\partial t} = \dfrac{\partial H_y}{\partial x} - \dfrac{\partial H_x}{\partial y} & \text{(vi)} & \mu \dfrac{\partial H_z}{\partial t} = -\dfrac{\partial E_y}{\partial x} + \dfrac{\partial E_x}{\partial y} \end{array} \tag{2.5}$$

It is clear from (2.5), (i)-(vi), that if $\vec{E}(x,y,0)$, $\vec{H}(x,y,0)$ are given, then the subsequent evolution of $E_z(x,y,t)$, $H_z(x,y,t)$ determine all of the other components. As for these components themselves, differentiating (2.5) (iii) and (2.5) (vi) with respect to t and then substituting (2.5) (iv), (v) and (2.5) (i), (ii) into the respectively resulting expressions, we obtain the familiar wave equations

$$\mu\epsilon \frac{\partial^2 E_z}{\partial t^2} = \frac{\partial^2 E_z}{\partial x^2} + \frac{\partial^2 E_z}{\partial y^2}, \tag{2.6}$$

$$\mu\epsilon \frac{\partial^2 H_z}{\partial t^2} = \frac{\partial^2 H_z}{\partial x^2} + \frac{\partial^2 H_z}{\partial y^2}, \tag{2.7}$$

valid for $(x,y) \in R$, $t \in [0,\infty)$, provided E_z, H_z have enough derivatives, or provided the equations are interpreted in the distributional sense. Assuming the initial states $\vec{E}(x,y,0)$, $\vec{H}(x,y,0)$ are divergence-free, we compute (cf. (2.4))

$$\epsilon \frac{\partial}{\partial t}\left(\frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y}\right) = \text{(using (2.5) (i), (ii))}$$

$$\epsilon\left(\frac{\partial^2 H_z}{\partial x \partial y} - \frac{\partial^2 H_z}{\partial y \partial x}\right) = 0$$

and similarly

$$\mu \frac{\partial}{\partial t}\left(\frac{\partial H_x}{\partial x} + \frac{\partial H_y}{\partial y}\right) = 0$$

and we conclude that the fields remain divergence-free for all time.

Suppose, then, that divergence-free initial states $\vec{E}(x,y,0)$, $\vec{H}(x,y,0)$ are given. Then $E_z(x,y,0)$, $H_z(x,y,0)$ are known and (2.5) (iii), (vi) determine $\frac{\partial E_z}{\partial t}(x,y,0)$ and $\frac{\partial H_z}{\partial t}(x,y,0)$. If (2.6), (2.7) are then solved with these initial conditions, and appropriate boundary conditions, the complete solution of Maxwell's equations (2.5) (i)-(vi), can be obtained by integrating (2.5) (i), (ii), (iv), (v). Thus it is enough to work with (2.6), (2.7), and it should be noted that the divergence condition does not have any bearing on E_z, H_z; it can be ignored henceforth.

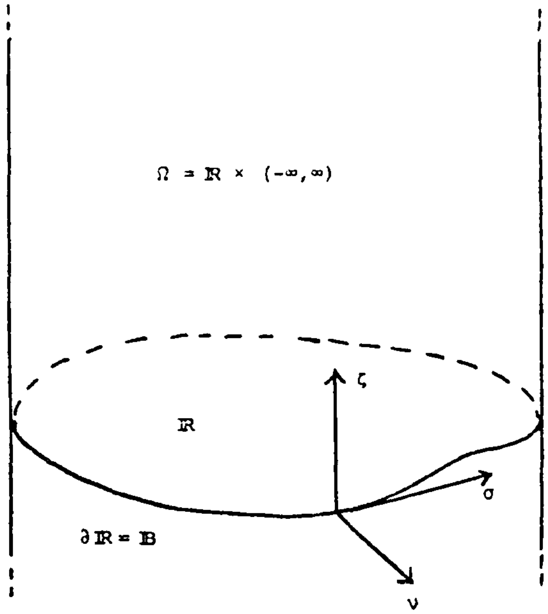

Figure 1. The Region $\mathbb{R}$

It is important to recast the boundary conditions (1.7), (1.8) so that they provide boundary conditions for (2.6), (2.7). We ask the reader to consult Figure 1, where the region R with boundary $\partial R = B$ is shown. At a point $(x,y) \in B$ we let $\vec{\nu} = \vec{\nu}(x,y)$ denote the unit exterior normal to B and we let $\vec{\sigma} = \vec{\sigma}(x,y)$ denote the positively oriented unit tangent vector to B there. With $\vec{\zeta}$, the unit vector in the positive z direction, $\vec{\nu}, \vec{\sigma}, \vec{\zeta}$ form a positively oriented orthogonal triple of unit vectors. Given an arbitrary vector w we can decompose it as

$$\vec{w} = (w_\nu, w_\sigma, w_\zeta(=w_z)) ,$$
$$|\vec{w}|^2 = w_\nu^2 + w_\sigma^2 + w_z^2 .$$

The tangential part of $\vec{H}$, which we have designated as $\vec{H}_\tau$, may now be represented as

$$\vec{H}_\tau = H_z\vec{\zeta} + H_\sigma\vec{\sigma} \tag{2.8}$$

and the current $\vec{J} = \vec{J}_\tau$ may likewise be represented as

$$\vec{J}_\tau = J_z\vec{\zeta} + J_\sigma\vec{\sigma} .$$

Then

$$\vec{\nu} \times \vec{J} = \vec{\nu} \times \vec{J}_{\tau} = \vec{\nu} \times (J_z \vec{\zeta} + J_{\sigma} \vec{\sigma}) = -J_z \vec{\sigma} + J_{\sigma} \vec{\zeta} \tag{2.9}$$

Combining (1.8), (2.8), (2.9) we see that on B

$$H_z(x,y,t) = J_{\sigma}(x,y,t) \,, \tag{2.10}$$

$$H_{\sigma}(x,y,t) = -J_z(x,y,t) \,. \tag{2.11}$$

Represent $\vec{\nu}$, $\vec{\sigma}$ as

$$\vec{\nu} = \nu_x \vec{\xi} + \nu_y \vec{\eta} \,, \tag{2.12}$$

$$\vec{\sigma} = \sigma_x \vec{\xi} + \sigma_y \vec{\eta} = -\nu_y \vec{\xi} + \nu_x \vec{\eta} \,. \tag{2.13}$$

Then compute

$$\frac{\partial E_z}{\partial \nu} = \frac{\partial E_z}{\partial x} \nu_x + \frac{\partial E_z}{\partial y} \nu_y = \text{(using (1.3)} \,, \tag{2.13}$$

$$= \mu \frac{\partial H_y}{\partial t} \sigma_y + \mu \frac{\partial H_x}{\partial t} \sigma_x = \mu \frac{\partial H_{\sigma}}{\partial t}$$

$$= \text{(using (2.11))} = -\frac{\partial J_z}{\partial t} \,. \tag{2.14}$$

The equations (2.10), (2.14) provide the needed boundary conditions for (2.6), (2.7) respectively. For H_z we have the Dirichlet-type boundary condition (2.10) while for E_z we have the Neumann-type boundary condition (2.14). If we let

$$\vec{U}(x,y,t) = \frac{\partial \vec{J}}{\partial t}(x,y,t) \,,$$

$$\vec{U} = \vec{U}_{\tau} = U_{\sigma} \vec{\sigma} + U_z \vec{\zeta} \,,$$

and differentiate (2.10), we have the more symmetric form

$$\frac{\partial H_z}{\partial t}(x,y,t) = U_{\sigma}(x,y,t), \quad \frac{\partial E_z}{\partial \nu} = -U_z(x,y,t), \; (x,y) \in B \,. \tag{2.15}$$

We complete this section by discussing the question of expression of the energy per unit cylinder length, (2.3), solely in terms of H_z and E_z.

We consider the equations (2.6), (2.7) with homogeneous boundary conditions

$$\frac{\partial H_z}{\partial t}(x,y,t) = 0, \quad \frac{\partial E_z}{\partial \nu}(x,y,t) = 0, \quad (x,y) \in B \,.$$

We use the symbol Δ for the Laplacian:

$$\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} .$$

Initially we take H_z, E_z to lie in the Sobolev space $H^2(R)$. This space must be decomposed in order to attach a meaning to Δ^{-1}.

The boundary condition for H_z may be rewritten as

$$H_z(x,y,t) = h(x,y), \quad (x,y) \in B ,$$

where, by the trace theorem, $h \in H^{3/2}(B)$. Then we can write

$$H_z(x,y,t) = \hat{H}_z(x,y,t) + \tilde{H}_z(x,y)$$

where $\tilde{H}_z(x,y)$ is the solution of

$$\Delta\tilde{H}_z(x,y) = 0, \quad \tilde{H}_z(x,y) = h(x,y), \quad (x,y) \in B$$

and

$$\hat{H}_z(x,y,t) = 0, \quad (x,y) \in B .$$

The inverse Laplacian Δ^{-1} is well defined on the functions $\hat{H}_z$. For E_z we may write

$$E_z(x,y,t) = \hat{E}_z(x,y,t) + \tilde{E}_z(t)$$

where $\tilde{E}_z$, as indicated, is constant with respect to $(x,y) \in R$ and

$$\int_B \hat{E}_z(x,y,t)ds = 0 .$$

It is well known that Δ^{-1} is well defined on the functions $\hat{E}_z$.

We proceed first on the assumption that

$$H_z(x,y,t) = \hat{H}_z(x,y,t), \quad E_z(x,y,t) = \hat{E}_z(x,y,t) .$$

We form new solutions of (2.6), (2.7) by setting

$$\mu \frac{\partial G_z}{\partial t} = -E_z, \quad \varepsilon \frac{\partial F_z}{\partial t} = H_z ,$$

$$G_z = \varepsilon\Delta^{-1} \frac{\partial E_z}{\partial t}, \quad F_z = \mu\Delta^{-1} \frac{\partial H_z}{\partial t} .$$

We then determine G_x, G_y, F_x, F_y, using the equations (2.5) with $\vec{G}$ replacing $\vec{H}$, $\vec{F}$ replacing $\vec{E}$, so that $\vec{F}$ and $\vec{G}$ satisfy Maxwell's equations:

$$\mu \frac{\partial \vec{G}}{\partial t} = -\text{curl } \vec{F} ,$$

$$\varepsilon \frac{\partial \vec{F}}{\partial t} = \text{curl } \vec{G} .$$

It will then be found that

$$\vec{E} = \text{curl } \vec{F}, \quad \vec{H} = \text{curl } \vec{G} .$$

Following this, (2.3) can be written as

$$\begin{aligned} E(t) &= \tfrac{1}{2}\iint_R (\varepsilon|\text{curl } \vec{F}|^2 + \mu|\text{curl } \vec{G}|^2)dxdy \\ &= \tfrac{1}{2}\iint_R \varepsilon\Big[\Big(\frac{\partial F_z}{\partial x}\Big)^2 + \Big(\frac{\partial F_z}{\partial y}\Big)^2 + \Big(\frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y}\Big)^2\Big] \\ &\quad + \mu\Big[\Big(\frac{\partial G_z}{\partial x}\Big)^2 + \Big(\frac{\partial G_z}{\partial y}\Big)^2 + \Big(\frac{\partial G_y}{\partial x} - \frac{\partial G_x}{\partial y}\Big)^2\Big]dxdy \end{aligned} \tag{2.16}$$

Then from (2.16) we have

$$\begin{aligned} E(t) &= \tfrac{1}{2}\iint_R \Big\{\varepsilon\Big[\Big(\frac{\partial F_z}{\partial x}\Big)^2 + \Big(\frac{\partial F_z}{\partial y}\Big)^2 + \Big(\mu \frac{\partial G_z}{\partial t}\Big)^2\Big] \\ &\quad + \mu\Big[\Big(\frac{\partial G_z}{\partial x}\Big)^2 + \Big(\frac{\partial G_z}{\partial y}\Big)^2 + \Big(\varepsilon \frac{\partial F_z}{\partial t}\Big)^2\Big]\Big\} dxdy \\ &= \tfrac{1}{2}\iint_R \varepsilon\Big[\Big(\frac{\partial F_z}{\partial x}\Big)^2 + \Big(\frac{\partial F_z}{\partial y}\Big)^2 + (E_z)^2\Big] \\ &\quad + \mu\Big[\Big(\frac{\partial G_z}{\partial x}\Big)^2 + \Big(\frac{\partial G_z}{\partial y}\Big)^2 + (H_z)^2\Big]dxdy \end{aligned}$$

Now consider the quadratic form (for $E_z = \hat{E}_z$)

$$\left(\frac{\partial E_z}{\partial t}, -\Delta^{-1}\frac{\partial E_z}{\partial t}\right) = \left(\mu\frac{\partial^2 G_z}{\partial t^2}, -\Delta^{-1}\frac{\partial^2 G_z}{\partial t^2}\right)$$

= (since G_z satisfies the wave equation $\mu\varepsilon\frac{\partial^2 G_z}{\partial t^2} = \Delta G_z$

and the boundary conditions $G_z(x,y,t) = 0$, $(x,y) \in B$) =

$$\frac{1}{\mu\varepsilon^2}(-\Delta G_z, G_z) = \frac{1}{\mu\varepsilon^2}\left[\left(\frac{\partial G_z}{\partial x}\right)^2 + \left(\frac{\partial G_z}{\partial y}\right)^2\right].$$

Similarly

$$\left(\frac{\partial H_z}{\partial t}, -\Delta^{-1}\frac{\partial H_z}{\partial t}\right) = \frac{1}{\varepsilon\mu^2}\left[\left(\frac{\partial F_z}{\partial x}\right)^2 + \left(\frac{\partial F_z}{\partial y}\right)^2\right]$$

from which it follows that

$$E(t) = \tfrac{1}{2}\iint_R \left\{(\mu\varepsilon)^2\left[\left(\frac{\partial E_z}{\partial t}, -\Delta^{-1}\frac{\partial E_z}{\partial t}\right) + \left(\frac{\partial H_z}{\partial t}, -\Delta^{-1}\frac{\partial H_z}{\partial t}\right)\right] + \varepsilon(E_z)^2 + \mu(H_z)^2\right\} dxdy .$$

I I finite states - a fact which will be very careful later.

It is necessary to modify this expression for general E_z, H_z. We begin with

$$E_z(x,y,t) = \tilde{E}_z(t) .$$

The only possible solutions of the wave equation (2.6) satisfying $\frac{\partial E_z}{\partial \nu}\Big|_B = 0$ and having this form are

$$E_z(x,y,t) = e_0 + e_1 t$$

where e_0 and e_1 are constants. (Such solutions are consistent with a constant boundary current J for which $J_\sigma \equiv 0$.) The corresponding E_x, E_y, H_z are zero but

$$\varepsilon e_1 = e\frac{\partial E_z}{\partial t} = \frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y} .$$

It is not possible to express this quantity in terms of E_z itself or H_z. It is better to leave it in the form $\varepsilon\frac{\partial E_z}{\partial t}$. Solutions of Maxwell's equations with E_z having this form have energy expressible as a quadratic form in E_z and $\frac{\partial E_z}{\partial t}$.

Next we consider $H_z = \tilde{H}_z$ as described earlier. Such a solution is consistent with a boundary current for which $J_\tau = 0$, constant with respect to time but possibly varying with $(x,y) \in B$. We may take H_x, H_y, E_z all zero. However,

$$\varepsilon \frac{\partial E_x}{\partial t} = \frac{\partial H_z}{\partial y}, \quad \varepsilon \frac{\partial E_y}{\partial t} = -\frac{\partial H_z}{\partial x}$$

so we may not assume that E_x and E_y are equal to zero. The energy associated with solutions of this type is expressible in terms of

$$\iint_R \left[\left(\frac{\partial H_z}{\partial x}\right)^2 + \left(\frac{\partial H_z}{\partial y}\right)^2\right] dxdy$$

if integration with respect to t is permitted. In the sequel we will not explicitly consider the timewise linear electric fields satisfying the above equations.

We see then that a norm involving only E_z and H_z and compatible with the energy (2.3) may be expressed as

$$\|(E_z,H_z)\|^2 = \iint_R (\mu\varepsilon)^2\left[\left(\frac{\partial \hat{E}_z}{\partial t}, -\Delta^{-1}\frac{\partial \hat{E}_z}{\partial t}\right) + \left(\frac{\partial \hat{H}_z}{\partial t}, -\Delta^{-1}\frac{\partial \hat{H}_z}{\partial t}\right)\right] + \varepsilon(\hat{E}_z)^2 + \mu(\hat{H}_z)^2$$

$$\rho_0(\tilde{E}_z)^2 + \rho_1\left(\frac{\partial \tilde{E}_z}{\partial t}\right)^2 + \sigma_0\left(\frac{\partial \tilde{H}_z}{\partial x}\right)^2 + \sigma_1\left(\frac{\partial \tilde{H}_z}{\partial y}\right)^2\Big]dxdy \qquad (2.17)$$

where $\rho_0, \rho_1, \sigma_0, \sigma_1$ are positive numbers. It will be seen that this is a weaker norm than the one associated with a pair of wave equations, viz.:

$$\|(E_z,H_z)\|^2 = \iint_R \left\{\mu\varepsilon\left[\left(\frac{\partial E_z}{\partial t}\right)^2 + \left(\frac{\partial H_z}{\partial t}\right)^2\right] + \|\nabla E_z\|^2 + \|\nabla H_z\|^2\right\}dxdy \ . \qquad (2.18)$$

We will denote the Hilbert space of states E_z, H_z, $\frac{\partial E_z}{\partial t}$, $\frac{\partial H_z}{\partial t}$ lying in $H^1(R)$, $H^1(R)$, $L^2(R)$, $L^2(R)$, respectively, by $\hat{H}$. This space will be very convenient for use in the remainder of this paper. In some cases we will add boundary conditions to the specification of H, the space with norm $\| \ \|$, without changing the symbol, to correspond to an agreed specification of the states in $\hat{H}$ by similar boundary conditions.

3. SOME CONTROL CONFIGURATIONS

We describe here two possible realizations of the control problem which we have posed and indicate why we have chosen the mathematically more interesting (i.e., more difficult) one to work with in this paper.

Let us assume that $\Gamma = \partial\Omega = B \times (-\infty,\infty)$ is covered by one or more layers of conducting bars, arranged in rows as shown in Figure 3.1. In the case of a single layer of conducting bars shown in Figure 2(b), the bars are arranged so that they make an angle θ, $0 < |\theta| < \frac{\pi}{2}$, with the vector $\vec{\sigma}$ (cf. Figure 1), while in the double layer case (Figure 2(a)) they are arranged so that the bars in the second layer make an angle ψ, $0 < |\psi| < \frac{\pi}{2}$, $\psi \neq \theta$, with the vector $\vec{\sigma}$. The current in any row of bars parallel to the z-axis is independent of z; i.e., constant for all bars in that row. As we consider successively smaller bars we obtain, as an idealization, the boundary current vector

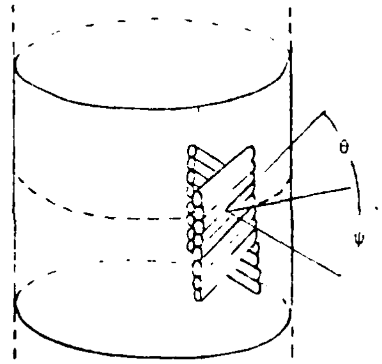

Figure 2(a). Double Layer Control

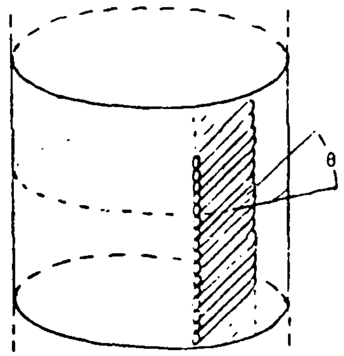

Figure 2(b). Single Layer Control

$$\vec{J}(x,y,t) = J(x,y,t)(\cos\theta\vec{\sigma} + \sin\theta\vec{\zeta}) \tag{3.1}$$

in the single layer case, $J(x,y,t)$ denoting the current strength with the sign determined so that J positive yields a positive current component in the $\vec{\sigma}$ direction. The corresponding formula in the double layer case is

$$\begin{aligned}\vec{J}(x,y,t) &= J_1(x,y,t)(\cos\theta\vec{\sigma} + \sin\theta\vec{\zeta}) \\ &\quad + J_2(x,y,t)(\cos\psi\vec{\sigma} + \sin\psi\vec{\zeta}) \ .\end{aligned} \tag{3.2}$$

The current components are, in the single layer case

$$J_\sigma(x,y,t) = J(x,y,t)\cos\theta \ ,$$

$$J_z(x,y,t) = J(x,y,t)\sin\theta \ ,$$

and in the double layer case,

$$\begin{pmatrix} J_\sigma(x,y,t) \\ J_z(x,y,t) \end{pmatrix} = \begin{pmatrix} \cos\theta & \cos\psi \\ \sin\theta & \sin\psi \end{pmatrix} \begin{pmatrix} J_1(x,y,t) \\ J_2(x,y,t) \end{pmatrix} . \tag{3.3}$$

The determinant of the matrix in (3.3) is $\sin(\psi - \theta) \neq 0$ if $\psi \neq \theta$ in the range $0 < |\theta| < \frac{\pi}{2}$, $0 < |\psi| < \frac{\pi}{2}$. Thus in the double layer case J_σ and J_z are independent if J_1 and J_2 are independent while in the single layer case J_σ and J_z are fixed non-zero multiples of each other.

The double layer case is easily disposed of in the light of earlier work on boundary control of the wave equation. Referring back to (2.10), (2.11) we now have, for $(x,y) \in B = \partial R$, $t \in [0,\infty)$,

$$\frac{\partial H_z}{\partial t}(x,y,t) = U_\sigma(x,y,t) = \cos\theta\, u_1(x,y,t) + \cos\psi\, u_2(x,y,t) ,$$

$$\frac{\partial E_z}{\partial \nu}(x,y,t) = -U_z(x,y,t) = -\sin\theta\, u_1(x,y,t) + \cos\psi\, u_2(x,y,t) ,$$

$$u_1(x,y,t) = \frac{\partial J_1}{\partial t}(x,y,t), \quad u_2(x,y,t) = \frac{\partial J_2}{\partial t}(x,y,t) .$$

Since U_σ and U_z are independent if u_1 and u_2 are, the control problem splits into two uncoupled wave-equation problems, one for E_z and one for H_z. These have been discussed thoroughly in [2], [3], [15], [16], [22], [23], [25] with affirmative controllability results for various control configurations and will not concern us further here.

In the remainder of this paper we study the single layer case. If we let

$$u(x,y,t) = \frac{\partial J}{\partial t}(x,y,t) \tag{3.4}$$

we now have the wave equations (2.6), (2.7) for E_z, H_z and the boundary conditions

$$\frac{\partial H_z}{\partial t}(x,y,t) = \cos\theta \frac{\partial J}{\partial t}(x,y,t) \equiv \alpha\, u(x,y,t) , \tag{3.5}$$

$$\frac{\partial E_z}{\partial \nu}(x,y,t) = -\sin\theta \frac{\partial J}{\partial t}(x,y,t) \equiv \beta\, u(x,y,t) . \tag{3.6}$$

The control problems for E_z and H_z are now coupled because the single control function, $u(x,y,t)$, appears in the boundary conditions for both E_z and H_z; we have to contol both systems simultaneously using the same control function.

If we rely on experience in a single space dimension, which has proved generally quite helpful in the control theory of a single wave equation, we are led to believe that systems like (2.6), (2.7), (3.5), (3.6) may, in fact, be controllable. Replacing $u(x,y,t)$ by $u_0(t)$, $u_1(t)$ and taking $0 < x < 1$, the one dimensional equations are, using variables v, w,

$$\rho \frac{\partial^2 v}{\partial t^2} - \frac{\partial^2 v}{\partial x^2} = 0 , \tag{3.7}$$

$$\frac{\partial v}{\partial t}(0,t) = \alpha u_0(t), \quad \frac{\partial v}{\partial t}(1,t) = \alpha u_1(t) , \tag{3.8}$$

$$\rho \frac{\partial^2 w}{\partial t^2} - \frac{\partial^2 w}{\partial x^2} = 0 \tag{3.9}$$

$$\frac{\partial w}{\partial x}(0,t) = -\beta u_0(t), \quad \frac{\partial w}{\partial x}(1,t) = \beta u_1(t) \tag{3.10}$$

(note that $-\frac{\partial w}{\partial x}$ corresponds to the exterior normal derivative at 0]. Letting

$$\tilde{v} = \frac{\partial v}{\partial x} \tag{3.11}$$

$$\tilde{w} = \frac{\partial w}{\partial t} \tag{3.12}$$

we find that

$$\rho \frac{\partial^2 \tilde{v}}{\partial t^2} - \frac{\partial^2 \tilde{v}}{\partial x^2} = 0 , \tag{3.13}$$

and

$$\rho \frac{\partial^2 \tilde{w}}{\partial t^2} - \frac{\partial^2 \tilde{w}}{\partial x^2} = 0 . \tag{3.14}$$

Differentiating (3.11) with respect to t and using (3.8) we have

$$\frac{1}{\rho}\frac{\partial^2 v}{\partial x^2}(0,t) = \frac{1}{\rho}\frac{\partial \tilde{v}}{\partial x}(0,t) = \frac{\alpha}{\rho} u_0'(t) , \tag{3.15}$$

$$\frac{1}{\rho}\frac{\partial^2 v}{\partial x^2}(1,t) = \frac{1}{\rho}\frac{\partial \tilde{v}}{\partial x}(1,t) = \frac{\alpha}{\rho} u_1'(t) , \tag{3.16}$$

while differentiation of (3.12) along with (3.10) yields

$$\frac{\partial^2 w}{\partial t \partial x}(0,t) = \frac{\partial \tilde{w}}{\partial x}(0,t) = -\beta u_0'(t) , \tag{3.17}$$

$$\frac{\partial^2 w}{\partial t \partial x}(1,t) = \frac{\partial \tilde{w}}{\partial x}(1,t) = \beta u_1'(t) . \tag{3.18}$$

Combining (3.13) with (3.14), (3.15), (3.16), (3.17), (3.18), we see that $\beta\tilde{v} + \frac{\alpha}{\rho}\tilde{w}$, $\beta\tilde{v} - \frac{\alpha}{\rho}\tilde{w}$ both satisfy the wave equation and

$$\frac{\partial}{\partial x}\left(\beta\tilde{v} + \frac{\alpha}{\rho}\tilde{w}\right)(0,t) = 0, \quad \frac{\partial}{\partial x}\left(\beta\tilde{v} + \frac{\alpha}{\rho}\tilde{w}\right)(1,t) = \frac{2\alpha\beta}{\rho} u_1'(1) ,$$

$$\frac{\partial}{\partial x}\left(\beta\tilde{v} - \frac{\alpha}{\rho}\tilde{w}\right)(0,t) = \frac{2\alpha\beta}{\rho} u_0'(t), \quad \frac{\partial}{\partial x}\left(\beta\tilde{v} - \frac{\alpha}{\rho}\tilde{w}\right)(1,t) = 0 .$$

Thus the control problems for $\beta\tilde{v} + \frac{\alpha}{\rho}\tilde{w}$ and $\beta\tilde{v} - \frac{\alpha}{\rho}\tilde{w}$ are both of Neumann type and are uncoupled. Affirmative controllability results are then available from [20], [21], [24].

If we replace $u_0(t)$ (or $u_1(t)$ by 0 in the above, then $\beta\tilde{v} - \frac{\alpha}{\rho}\tilde{w}$ (or $\beta\tilde{v} + \frac{\alpha}{\rho}\tilde{w}$) will become completely uncontrollable and our original system must therefore be uncontrollable. This result at first seems to predict failure for the enterprize which we now undertake for the two dimensional case.

4. APPROXIMATE BOUNDARY CONTROLLABILITY

By a simple change of scale in the t variable, and renaming of the independent variables, we may assume that the system of interest is

$$\left.\begin{aligned} \frac{\partial^2 v}{\partial t^2} &= \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}, \qquad (4.1) \\ \frac{\partial^2 w}{\partial t^2} &= \frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2}, \qquad (4.2) \end{aligned}\right\} \quad \begin{aligned} & t > 0 \\ & (x,y) \in R, \end{aligned}$$

with boundary conditions

$$\left.\begin{aligned} \frac{\partial v}{\partial t}(x,y,t) &= \alpha u(x,y,t) \qquad (4.3) \\ \frac{\partial w}{\partial \nu}(x,y,t) &= \beta u(x,y,t) \qquad (4.4) \end{aligned}\right\} \quad \begin{aligned} & t > 0, \\ & (x,y) \in B = \partial\Omega \end{aligned}$$

We will not, in general, assume that $u(x,y,t)$ can be selected at will for all values of (x,y,t) shown. More on this later.

Because the system is time reversible, it is sufficient to analyze controllability in terms of control from the zero initial state

$$\left.\begin{aligned} v(x,y,0) &= \frac{\partial v}{\partial t}(x,y,0) = 0, \qquad (4.5) \\ w(x,y,0) &= \frac{\partial w}{\partial t}(x,y,0) = 0, \qquad (4.6) \end{aligned}\right\} \quad (x,y) \in R,$$

to a final state

$$\left.\begin{aligned} v(x,y,T) &= v_0(x,y), \quad \frac{\partial v}{\partial t}(x,y,T) = v_1(x,y) \qquad (4.7) \\ w(x,y,T) &= w_0(x,y), \quad \frac{\partial w}{\partial t}(x,y,T) = w_1(x,y) \qquad (4.8) \end{aligned}\right\} \quad (x,y) \in R.$$

We have noted in Section 2 that the $\| \ \|$- finite states are dense in the $| \ |$- finite states. In the present context this means that we can work with the Hilbert space of states v, $\frac{\partial v}{\partial t}$, w, $\frac{\partial w}{\partial t}$ with the inner product

$$\left(\left(v, \frac{\partial v}{\partial t}, w, \frac{\partial w}{\partial t}\right); \left(\tilde{v}, \frac{\partial \tilde{v}}{\partial t}, \tilde{w}, \frac{\partial \tilde{w}}{\partial t}\right)\right)$$

$$= \iint_R \left[\frac{\partial v}{\partial t}\frac{\partial \tilde{v}}{\partial t} + \frac{\partial w}{\partial t}\frac{\partial \tilde{w}}{\partial t} + \frac{\partial v}{\partial x}\frac{\partial \tilde{v}}{\partial x} + \frac{\partial w}{\partial x}\frac{\partial \tilde{w}}{\partial x} + \frac{\partial v}{\partial y}\frac{\partial \tilde{v}}{\partial y} + \frac{\partial w}{\partial y}\frac{\partial \tilde{w}}{\partial y}\right] dxdy \ , \qquad (4.9)$$

a space which we will refer to as $\hat{H}$. The norm is $\| \ \|$ (cf. (2.18)) with $uc = 1$. As we have indicated, this is a dense subspace of H, the Hilbert space obtained by use of the norm $| \ |$ (cf. (2.17)).

The final states (4.7), (4.8) are not quite arbitrary in $\hat{H}$ if the control u is restricted so that its support is contained in a proper relatively closed subset $B_1 \subset B$. Since the condition

$$\frac{\partial v}{\partial t}(x,y,t) = \alpha\, u(x,y,t), \quad (x,y) \in B$$

applies, we may as well adjoin the additional condition

$$v_0(x,y) = 0, \quad (x,y) \in B - B_1 \equiv B_0 \ . \qquad (4.10)$$

The trace theorem ([1], [19]) assures us that this describes a closed subspace of $\hat{H}$, which we will call $\hat{H}_1$. The only restriction on $\hat{H}_1$ is (4.10); v_0 is permitted to have arbitrary values in $H^{1/2}(B_1)$ and w_0, w_1 are unrestricted in $H^1(B)$, $H^0(R) = L^2(R)$, respectively.

Let U be a given space of admissible control functions, about which we will shortly have more to say. For each control $u \in U$ we assume the existence of a unique solution v_u, w_u of (4.1)-(4.6) for $t > 0$, $(x,y) \in R$. Very general sufficient conditions for this to be the case are given in [19]. We define the reachable set at time T, $R(U,T)$, to be the set of all final states $v_u(x,y,T)$, $\frac{\partial v_u}{\partial t}(x,y,T)$, $w_u(x,y,T)$, $\frac{\partial w_u}{\partial t}(x,y,T)$ which may be realized in this way. The set $R(U,T)$ is a subspace of $\hat{H}_1$ if U is a linear space, which we will assume, and our system is approximately controllable in time T if $R(U,T)$ is dense in $\hat{H}_1$ (then $R(U,T)$ is also dense in H because $| \ |$ is a weaker norm than $\| \ \|$ and $\hat{H}_1$ is dense in H). Evidently $R(U,T)$ is dense in $\hat{H}_1$ just in case, given an arbitrary state $(\tilde{v}_0,\tilde{v}_1,\tilde{w}_0,\tilde{w}_1)$ in $\hat{H}_1$,

$$\Big\{\Big(\Big(v_u(x,y,T),\ \frac{\partial v_u}{\partial t}(x,y,T),\ w_u(x,y,T),\ \frac{\partial w_u}{\partial t}(x,y,T)\Big);(\tilde{v}_0,\tilde{v}_1,\tilde{w}_0,\tilde{w}_1)\Big) = 0\ ,$$

$$u \in U\Big\} \Longrightarrow (\tilde{v}_0,\tilde{v}_1,\tilde{w}_0,\tilde{w}_1) = 0\ . \tag{4.11}$$

Let $\tilde{v}(x,y,t)$, $\tilde{w}(x,y,t)$ be the unique solution of (4.1), (4.2) satisfying the terminal conditions at time T:

$$\tilde{v}(x,y,T) = \tilde{v}_0,\ \frac{\partial \tilde{v}}{\partial t}(x,y,T) = \tilde{v}_1,\ \tilde{w}(x,y,T) = \tilde{w}_0,\ \frac{\partial \tilde{w}}{\partial t}(x,y,T) = \tilde{w}_1\ , \tag{4.12}$$

and the homogeneous boundary conditions

$$\frac{\partial \tilde{v}}{\partial t}(x,y,t) = 0\ , \tag{4.13}$$

$$\frac{\partial \tilde{w}}{\partial \nu}(x,y,t) = 0\ , \tag{4.14}$$

$(x,y) \in B,\quad t > 0\ .$

Computing the quantity

$$\frac{d}{dt}\Big(\Big(v_u(x,y,t),\ \frac{\partial v_u}{\partial t}(x,y,t),\ w_u(x,y,t),\ \frac{\partial w_u}{\partial t}(x,y,t)\Big)\ ;$$

$$\Big(\tilde{v}(x,y,t),\ \frac{\partial \tilde{v}}{\partial t}(x,y,t),\ \tilde{w}(x,y,t),\ \frac{\partial \tilde{w}}{\partial t}(x,y,t)\Big)\Big)\ ,$$

using familiar duality theorems involving the Laplacian and integrating from 0 to T (see [22], [23], [26] for details in the case of a single wave equation) we see that

$$\Big(\Big(v_u(x,y,T),\ \frac{\partial v_u}{\partial t}(x,y,T),\ w_u(x,y,T),\ \frac{\partial w_u}{\partial t}(x,y,T)\Big);\ (\tilde{v}_0,\tilde{v}_1,\tilde{w}_0,\tilde{w}_1)\Big)$$

$$= \int_0^T \int_B \Big[\frac{\partial \tilde{v}}{\partial t}(x,y,t)\ \frac{\partial v_u}{\partial \nu}(x,y,t) + \frac{\partial \tilde{v}}{\partial \nu}(x,y,t)\ \frac{\partial v_u}{\partial t}(x,y,t)$$

$$+ \frac{\partial \tilde{w}}{\partial t}(x,y,t)\ \frac{\partial w_u}{\partial \nu}(x,y,t) + \frac{\partial \tilde{w}}{\partial \nu}(x,y,t)\ \frac{\partial w_u}{\partial t}(x,y,t)\Big]ds\,dt\ . \tag{4.15}$$

Then using the boundary conditions (4.3), (4.4), (4.13), (4.14) we see that the above reduces to

$$\int_0^T \int_B \left[\alpha \frac{\partial \tilde{v}}{\partial \nu}(x,y,t) + \beta \frac{\partial \tilde{w}}{\partial t}(x,y,t)\right] u(x,y,t)\,ds\,dt \;. \tag{4.16}$$

If, as discussed above, we suppose that B has the disjoint decomposition

$$B = B_0 \cup B_1 \;,$$

with B_1 relatively open in B, and that $u(x,y,t) \equiv 0$, $(x,y) \in B_0$ while on B_1 u is unrestricted save for the specification of the admissible space (e.g., we might take

$$U = C(B_1 \times [0,T]), \quad U = L^2(B_1 \times [0,T]) \;, \tag{4.17}$$

or any of many other possibilities), and if we suppose the first equation in (4.11) to hold, we conclude that (4.16) vanishes for all $u \in U$. We know from the trace theorem ([1], [19]) that the partial derivatives

$$\frac{\partial \tilde{v}}{\partial t}, \frac{\partial \tilde{v}}{\partial \nu}, \frac{\partial \tilde{w}}{\partial t}, \frac{\partial \tilde{w}}{\partial \nu} \;,$$

restricted to B, all lie in $H^{1/2}(B)$ for $t \in [0,T]$ and vary, with respect to the norm in that space, continuously with respect to t, i.e. they lie in $C(H^{1/2}(B); [0,T])$. We suppose, as is the case for (4.17), e.g., that U includes a total subspace of the dual space of $C(H^{1/2}(B_1); [0,T])$. Then the fact that (4.17) is zero for all $u \in U$ implies

$$\alpha \frac{\partial \tilde{v}}{\partial \nu}(x,y,t) + \beta \frac{\partial \tilde{w}}{\partial t}(x,y,t) = 0, \quad (x,y) \in B_1, \quad t \in [0,T] \;. \tag{4.18}$$

We also have (cf. (4.13), (4.14))

$$\frac{\partial \tilde{v}}{\partial t}(x,y,t) = 0, \quad \frac{\partial \tilde{w}}{\partial \nu}(x,y,t) = 0, \quad (x,y) \in B_1, \quad t \in [0,T] \;. \tag{4.19}$$

The boundary values of $\tilde{v}$ and $\tilde{w}$ are therefore <u>overspecified</u> on $B_1 \times [0,T]$. The proof of approximate controllability, where it can be carried through, depends upon being able to use this overspecification to show that

$$\tilde{v}(x,y,t) \equiv 0, \quad \tilde{w}(x,y,t) \equiv 0, \quad (x,y) \in R, \quad t \in [0,T] \;,$$

and therefore to conclude that the implication (4.11) is indeed valid so that $R(U,T)$ is dense in $\hat{H}_1$ and hence in H. We carry this argument out for the case in which R is a rectangle and B_1 is one of its sides in Section 5.

Following the development in [6], it may be seen that our system is exactly controllable in $\hat{H}_1$, using the control space $U = L^2(B_1 \times [0,T])$, just in case

$$\left\| \alpha \frac{\partial \tilde{v}}{\partial \nu} + \beta \frac{\partial \tilde{w}}{\partial t} \right\|_{L^2(B_1 \times [0,T])} \geq K \left\| (\tilde{v}_0, \tilde{v}_1, \tilde{w}_0, \tilde{w}_1) \right\|_{\hat{H}} \tag{4.20}$$

for some $K > 0$. In general this is a very difficult result to obtain but we are able to obtain exact controllability, by other means, for the case where **R** is a disc in R^2 and $B_1 = B$ is its boundary, a circle. This result is developed in Section 6 where it will be seen that it is heavily dependent on certain properties of the Bessel functions.

5. THE CASE R = A RECTANGLE, B_1 = ONE SIDE.

The work here can be carried out for a rectangle with arbitrary dimensions, but all essential ideas are contained in the notationally simpler case

$$R = \{(x,y) | 0 < x < \pi,\ 0 < y < \pi\}$$

to which attention is restricted henceforth. We will assume that B_1, the portion of the boundary on which control is exercised, is one side of R, without loss of generality it is the set

$$B_1 = \{(\pi,y) | 0 < y < \pi\} \ . \tag{5.1}$$

We consider then $\tilde{v}$, $\tilde{w}$ satisfying (4.1), (4.2) in $R \times [0,T]$ for some $T > 0$, and also satisfying boundary conditions

$$\frac{\partial \tilde{v}}{\partial t}(x,y,t) = 0,\ \frac{\partial \tilde{w}}{\partial \nu}(x,y,t) = 0,\ (x,y) \in B = \partial R,\ t \in [0,T] \ , \tag{5.2}$$

$$\alpha \frac{\partial \tilde{v}}{\partial \nu}(\pi,y,t) + \beta \frac{\partial \tilde{w}}{\partial t}(\pi,y,t)$$

$$= \alpha \frac{\partial \tilde{v}}{\partial x}(\pi,y,t) + \beta \frac{\partial \tilde{w}}{\partial t}(\pi,y,t) = 0,\quad 0 < y < \pi,\quad t \in [0,T] \ . \tag{5.3}$$

We may assume without loss of generality, since the wave equation is time reversible with either Dirichlet or Neumann boundary conditions, that $\tilde{v}$ and $\tilde{w}$ are extended to satisfy (4.1), (4.2) on $-\infty < t < \infty$ and that the boundary conditions (5.2) hold for $(x,y) \in B$, $t \in (-\infty,\infty)$. We may not assume that the boundary condition (5.3) is applicable beyond $[0,T]$, however, if controls are restricted to have support in $B_1 \times [0,T]$. Let $\delta > 0$ and let $s(t)$ be an arbitrary function in $C^\infty(-\infty,\infty)$ with support in $(-\delta,\delta)$. Define

$$\hat{v}(x,y,t) = \int_{-\infty}^{\infty} s(t-\tau)\tilde{v}(x,y,\tau)d\tau \ , \tag{5.4}$$

$$\hat{w}(x,y,t) = \int_{-\infty}^{\infty} s(t-\tau)\tilde{w}(x,y,\tau)d\tau \ . \tag{5.5}$$

Then $\hat{v}$, $\hat{w}$ are solutions of the wave equations (4.1), (4.2) satisfying boundary conditions

$$\frac{\partial \hat{v}}{\partial t}(x,y,t) = 0, \quad \frac{\partial \hat{w}}{\partial \nu}(x,y,t) = 0, \quad (x,y) \in B = \partial R, \quad -\infty < t < \infty , \tag{5.6}$$

while

$$\alpha \frac{\partial \hat{v}}{\partial x}(\pi,y,t) + \beta \frac{\partial \hat{w}}{\partial t}(\pi,y,t) = 0, \quad 0 < y < \pi, \quad t \in [\delta, T - \delta] . \tag{5.7}$$

Moreover, it can be shown that $\hat{v}$, $\hat{w}$ are of class C^∞ for $(x,y) \in R$, $-\infty < t < \infty$. If we can show $\hat{v} \equiv 0$, $\hat{w} \equiv 0$ for any such choice of s, then $\tilde{v} \equiv 0$, $\tilde{w} \equiv 0$.

Let us define, for $(x,y) \in R$, $-\infty < t < \infty$,

$$\phi(x,y,t) = \alpha \frac{\partial \hat{v}}{\partial x}(x,y,t) + \beta \frac{\partial \hat{w}}{\partial t}(x,y,t) . \tag{5.8}$$

From (5.7) we have

$$\phi(\pi,y,t) = 0, \quad 0 < y < \pi, \quad t \in [\delta, T - \delta] . \tag{5.9}$$

Since α and β are constants we have

$$\frac{\partial^2 \phi}{\partial t^2} = \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2}, \quad (x,y) \in R, \quad -\infty < t < \infty . \tag{5.10}$$

Let us note that, since $\hat{v}$ satisfies the wave equation in $R \cup B$,

$$\begin{aligned} &\alpha \frac{\partial^2 \hat{v}}{\partial t^2}(x,y,t) + \beta \frac{\partial^2 \hat{w}}{\partial t \partial x}(x,y,t) \\ &= \alpha\left[\frac{\partial^2 \hat{v}}{\partial x^2}(x,y,t) + \frac{\partial^2 \hat{v}}{\partial y^2}(x,y,t)\right] + \beta \frac{\partial^2 \hat{w}}{\partial t \partial x}(x,y,t) . \end{aligned} \tag{5.11}$$

Setting $x = \pi$ in (5.11) and differentiating the identities in (5.6) with respect to t, we see that the left hand side vanishes. Then, comparing (5.11) with (5.8)

$$\frac{\partial \phi}{\partial x}(\pi,y,t) = -\alpha \frac{\partial^2 \hat{v}}{\partial y^2}(\pi,y,t) \equiv \alpha(y), \quad 0 < y < \pi, \quad \delta < t < T - \delta , \tag{5.12}$$

the last identity being valid as a consequence of the first condition in (5.6).

The two conditions, (5.8) and (5.12), satisfied by ϕ at the boundary $x = \pi$ enable us to use Holmgren's uniqueness theorem (see [5] or [13], e.g.) in much the same way as it was used in the proof of the approximate controllability of the wave equation in [22], [23] to see that if

$$T > 2 + 2\delta \tag{5.13}$$

then ϕ must be independent of t for $1 + \delta < t < T - 1 - \delta$, i.e.

$$\phi(x,y,t) = \phi(x,y), \quad (x,y) \in R, \quad 1 + \delta < t < T - 1 - \delta. \tag{5.14}$$

Because $\hat{v}$ and $\hat{w}$ satisfy the wave equation in R with the homogeneous boundary conditions (5.6), and are of class C^∞ in $R \cup B$, we have C^∞- convergent expansions

$$\hat{v}(x,y,t) = \hat{v}_0(x,y) + \sum_{k=1}^{\infty}\sum_{j=1}^{\infty}\left(v_{kj}e^{i\omega_{kj}t} + \bar{v}_{kj}e^{-\omega_{kj}t}\right)\sin kx \sin jy, \tag{5.15}$$

$$\hat{w}(x,y,t) = \hat{w}_0 + \sum_{k=1}^{\infty}\sum_{j=1}\left(w_{kj}e^{i\omega_{kj}t} + \bar{w}_{kj}e^{-i\omega_{kj}t}\right)\cos kx \cos jy, \tag{5.16}$$

where

$$w_{kj} = \sqrt{k^2 + j^2}, \tag{5.17}$$

$\hat{v}_0(x,y)$ is a C^∞ function in $R \cup B$ such that (cf. (4.10))

$$\hat{v}_0(x,y) = 0, \quad (x,y) \in B - \{(\pi,y) \mid 0 < y < \pi\} \tag{5.18}$$

and $\hat{w}_0$ is a constant. Then, from (5.8),

$$\phi(x,y,t) - \alpha\frac{\partial\hat{v}_0(x,y)}{\partial x} =$$

$$= \sum_{k=1}^{\infty}\cos kx\Big[\sum_{j=1}^{\infty}(\alpha k v_{kj}\sin jy + i\beta\omega_{kj}w_{kj}\cos jy)e^{i\omega_{kj}t}$$

$$+ \sum_{j=1}^{\infty}(\alpha k\bar{v}_{kj}\sin jy - i\beta\omega_{kj}\bar{w}_{kj}\cos jy)e^{-i\omega_{kj}t}\Big], \tag{5.19}$$

still C^∞- convergent for $(x,y) \in R \cup B$, $-\infty < t < \infty$. Noting (5.14), we see that the left hand side takes the form

$$\phi(x,y,t) - \alpha\frac{\partial\hat{v}_0(x,y)}{\partial x} = \phi(x,y) - \alpha\frac{\partial\hat{v}_0(x,y)}{\partial x} \equiv \hat{\phi}(x,y),$$

$$1 + \delta < t < T - 1 - \delta. \tag{5.20}$$

We now strengthen (5.13) to

$$T > 4 + 2\delta \tag{5.21}$$

and we see that the time interval in (5.14), (5.20) has length > 2, i.e.

$$T - 1 - \delta - (1 + \delta) = T - (2 + 2\delta) > 2 . \tag{5.17}$$

Since the functions $\sqrt{\frac{2}{\pi}} \cos kx$ are orthonormal on $0 < x < \pi$, we conclude from (5.19), (5.20) that for $k = 1,2,3,...$

$$\sum_{j=1}^{\infty} (\alpha k v_{kj} \sin jy + i\beta\omega_{kj} w_{kj} \cos jy) e^{i\omega_{kj} t}$$

$$+ \sum_{j=1}^{\infty} (\alpha k \bar{v}_{kj} \sin jy - i\beta\omega_{kj} \bar{w}_{kj} \cos jy) e^{-i\omega_{kj} t}$$

$$= \frac{2}{\pi} \int_0^{\pi} \hat{\phi}(x,y) \cos kx \, dx \equiv \Phi_k(y), \quad 1 + \delta < t < T - 1 - \delta . \tag{5.22}$$

Classical results of Levinson and Schwartz ([17], [27]), which have frequently been used in control studies of this type (see, e.g., [12], [21]), can now be used to show that for each fixed k, the exponential functions

$$e^{\pm i\omega_{kj} t} = e^{\pm i\sqrt{k^2+j^2}\, t}, \qquad j = 1,2,3,... ,$$

together with the constant function 1 are strongly independent in $L^2(I)$ for any t-interval I of length > 2. This clearly contradicts (5.22) unless we have

$$\Phi_k(y) \equiv 0, \quad 0 < y < \pi \tag{5.23}$$

and

$$\alpha k v_{kj} \sin jy + i\beta\omega_{kj} w_{kj} \cos jy = 0, \quad 0 < y < \pi, \quad j = 1,2,3,... .$$

But then, since for each j $\sin jy$ and $\cos jy$ are independent on $0 < y < \pi$ and since none of α, k, β, ω_{kj} are zero, we conclude that

$$v_{kj} = 0, \quad w_{kj} = 0, \quad k = 1,2,3,...., \quad j = 1,2,3,... . \tag{5.24}$$

Since (5.22), (5.23) show that

$$\hat{\phi}(x,y) = \sum_{k=1}^{\infty} \Phi_k(y) \cos kx = 0 ,$$

(5.19) gives

$$\phi(x,y,t) = \phi(x,y) = \alpha \frac{\partial \hat{v}_0(x,y)}{\partial x}, \quad (x,y) \in R, \tag{5.25}$$
$$1 + \delta \leq t \leq T - 1 - \delta .$$

Noting (5.15) and (5.16) and the fact that $\hat{v}(0,y,t) \equiv 0$, we conclude from (5.23) that

$$\left.\begin{aligned} \hat{v}(x,y,t) &\equiv \hat{v}_0(x,y) , \\ \hat{w}(x,y,t) &\equiv \hat{w}_0 , \end{aligned}\right\} \quad 1 + \delta \leq t \leq T - 1 - \delta . \tag{5.26}$$

Since $v(x,y,t) \equiv v_0(x,y)$ is a solution of the wave equation with (cf. (5.18))

$$\hat{v}_0(x,y) = 0, \quad (x,y) \in B - \{(\pi,y) | 0 \leq y \leq \pi\}$$

it must in fact be a solution of Laplace's equation there. Then we compute

$$\int_R \left[\left(\frac{\partial \hat{v}_0}{\partial x}(x,y)\right)^2 + \left(\frac{\partial \hat{v}_0}{\partial y}(x,y)\right)^2 + \hat{v}_0(x,y)\left(\frac{\partial^2 \hat{v}_0}{\partial x^2}(x,y) + \frac{\partial^2 \hat{v}_0}{\partial y^2}(x,y)\right)\right]dxdy$$

$$= \int_R \operatorname{div}(\hat{v}_0(x,y)\operatorname{grad} \hat{v}_0(x,y))dxdy$$

$$= \int_B \hat{v}_0(x,y)\operatorname{grad} \hat{v}_0(x,y)\cdot\nu(x,y)ds = \int_0^\pi \hat{v}_0(\pi,y) \frac{\partial \hat{v}_0}{\partial x}(\pi,y)dy . \tag{5.27}$$

Combining (5.9) and (5.25) with the fact that $\hat{v}_0$ satisfies Laplace's equation we conclude from(5.27) that

$$\int_R \left[\left(\frac{\partial \hat{v}_0}{\partial x}(x,y)\right)^2 + \left(\frac{\partial \hat{v}_0}{\partial y}(x,y)\right)^2\right]dxdy = 0$$

and this, together with (5.18), implies

$$\hat{v}_0(x,y) \equiv 0 . \tag{5.28}$$

Combining (5.26) and (5.28) we conclude that

$$\left.\begin{aligned} \hat{v}(x,y,t) &\equiv 0 \\ \hat{w}(x,y,t) &\equiv \hat{w}_0 \end{aligned}\right\} \quad (x,y) \in R, \quad -\infty < t < \infty \tag{5.29}$$

the result for $-\infty < t < \infty$ being an immediate consequence of the result for

$1 + \delta \le t \le T - 1 - \delta$. Since this is true for every $\delta > 0$ and every $s(t)$ in (5.4), (5.5), we conclude that a comparable result obtains for $\hat{v}$, $\hat{w}$ in (4.11), (5.2), (5.3). It follows (since w = constant is a zero state in $\hat{H}$ and in H) that (cf. (4.9) ff.)

$$\|(\tilde{v}_0, \tilde{v}_1, \tilde{w}_0, \tilde{w}_1)\|_{\hat{H}} = \|(\tilde{v}_0, \tilde{v}_1, \tilde{w}_0, \tilde{w}_1)\|_H = 0$$

and, from the discussion in Section 4, the approximate controllability result follows.

6. SOME EXACT CONTROLLABILITY RESULTS IN THE CASE OF A CIRCULAR CYLINDER

We consider now the case $\Omega = R \times (-\infty,\infty)$ with

$$R = \{(x,y) \mid x^2 + y^2 < 1\} ,$$

$$B = \partial R = \{(x,y) \mid x^2 + y^2 = 1\} .$$

With introduction of the usual polar coordinates r,θ, the equations (4.1), (4.2) now become

$$\frac{\partial^2 v}{\partial t^2} = \frac{\partial^2 v}{\partial r^2} + \frac{1}{r}\frac{\partial v}{\partial r} + \frac{1}{r^2}\frac{\partial^2 v}{\partial \theta^2} \tag{6.1}$$

$$\frac{\partial^2 w}{\partial t^2} = \frac{\partial^2 w}{\partial r^2} + \frac{1}{r}\frac{\partial w}{\partial r} + \frac{1}{r^2}\frac{\partial^2 w}{\partial \theta^2} \tag{6.2}$$

and the boundary conditions (4.3), (4.4) are transformed to

$$\frac{\partial v}{\partial t}(1,\theta,t) = \alpha u(\theta,t) , \tag{6.3}$$

$$\frac{\partial w}{\partial r}(1,\theta,t) = \beta u(\theta,t) . \tag{6.4}$$

Writing

$$v(r,\theta,t) = \sum_{k=-\infty}^{\infty} v_k(r,t)e^{ik\theta}, \quad v_{-k} = \bar{v}_k , \tag{6.5}$$

$$w(r,\theta,t) = \sum_{k=-\infty}^{\infty} w_k(r,t)e^{ik\theta}, \quad w_{-k} = \bar{w}_k , \tag{6.6}$$

$$u(\theta,t) = \sum_{k=-\infty}^{\infty} u_k(t)e^{ik\theta} \tag{6.7}$$

we arrive at an infinite collection of control problems in the single space dimension, r:

$$\frac{\partial^2 v_k}{\partial t^2} = \frac{\partial^2 v_k}{\partial r^2} + \frac{1}{r}\frac{\partial v_k}{\partial r} - \frac{k^2}{r^2}\cdot v_k = 0, \qquad -\infty < k < \infty , \tag{6.8}$$

$$\frac{\partial^2 w_k}{\partial t^2} = \frac{\partial^2 w_k}{\partial r^2} + \frac{1}{r}\frac{\partial w_k}{\partial r} - \frac{k^2}{r^2} w_k = 0, \qquad -\infty < k < \infty , \tag{6.9}$$

$$\frac{\partial v_k}{\partial t}(1,t) = \alpha u_k(t), \qquad -\infty < k < \infty, \tag{6.10}$$

$$\frac{\partial w_k}{\partial r}(1,t) = \beta u_k(t), \qquad -\infty < k < \infty. \tag{6.11}$$

We will first treat the equation (4.1) with the boundary condition (4.3) which, as we have seen, reduces to the set of problems (6.8), (6.10), $-\infty < k < \infty$. With

$$z(r,\theta,t) = \sum_{k=-\infty}^{\infty} z_k(r,t)e^{ik\theta} = \sum_{k=-\infty}^{\infty} \frac{\partial v_k(r,t)}{\partial t} e^{ik\theta} = \frac{\partial v}{\partial t}(r,\theta,t)$$

we have the equivalent first order systems

$$\frac{\partial}{\partial t}\begin{pmatrix} v_k(r,t) \\ z_k(r,t) \end{pmatrix} = \begin{pmatrix} 0 & I \\ L_{|k|} & 0 \end{pmatrix}\begin{pmatrix} v_k(r,t) \\ z_k(r,t) \end{pmatrix} = \mathbf{L}_{|k|}\begin{pmatrix} v_k(r,t) \\ z_k(r,t) \end{pmatrix} \tag{6.12}$$

where $L_{|k|}$ is the differential operator on the right hand side of (6.8). The boundary conditions (6.10) become

$$z_k(1,t) = \alpha u_k(t), \quad -\infty < k < \infty. \tag{6.13}$$

The eigenvalues of the operator $\mathbf{L}_{|k|}$ with the corresponding homogeneous boundary condition

$$z_k(1,t) = 0 \tag{6.14}$$

are

$$0, \quad \pm i\omega_{|k|,\ell}, \qquad \ell = 1,2,3,\ldots,$$

where $\omega_{|k|,\ell}$ is the ℓ-th positive zero of the Bessel function $J_{|k|}(r)$ of order $|k|$. The corresponding vector eigenfunctions are

$$\begin{pmatrix} \phi_{|k|0}(r) \\ 0 \end{pmatrix}, \begin{pmatrix} \phi_{|k|\ell}(r) \\ \pm i\omega_{|k|,\ell}\,\phi_{|k|\ell}(r) \end{pmatrix}, \quad \begin{matrix} -\infty < k < \infty \\ \ell = 1,2,3,\ldots; \end{matrix}$$

where

$$\phi_{|k|,0}(r) = A_{|k|,0}\, r^{|k|}, \qquad -\infty < k < \infty, \tag{6.15}$$

$$\phi_{|k|\ell}(r) = A_{|k|\ell} J_{|k|}(\omega_{|k|,\ell} r), \qquad \begin{array}{l} -\infty < k < \infty , \\ \ell = 1,2,3,\ldots . \end{array}$$

The normalization coefficients $A_{|k|,0}$, $A_{|k|,\ell}$ are chosen so that

$$\int_0^1 r|\phi_{|k|,0}(r)|^2 dr = \frac{1}{2\pi}, \int_0^1 r|\phi_{|k|,\ell}(r)|^2 dr = \frac{1}{2\pi}, \quad \ell = 1,2,3,\ldots . \tag{6.16}$$

Thus

$$A_{|k|,0} = \sqrt{\frac{|k| + 1}{\pi}}, \qquad -\infty < k < \infty , \tag{6.17}$$

while, as may be seen from [5], e.g.

$$A_{|k|,\ell} = \frac{\omega_{|k|,\ell}}{\sqrt{\pi}\, J'_{|k|,\ell}(\omega_{|k|,\ell})} \tag{6.18}$$

The state space in which we wish to work, for the present at least, is (cf. (2.18))

$$\tilde{H} = \left\{ \binom{v}{z} \middle| v \in H^1(R),\ z \in L^2(R) \right\}$$

with the inner product

$$\left(\binom{v_1}{z_1}, \binom{v_2}{z_2} \right) = \int_R (\nabla v_1 \cdot \overline{\nabla v_2} + z_1 \overline{z_2}) dxdy$$

and associated norm. Since the $\phi_{|k|,\ell}$ satisfy the homogeneous boundary condition (6.14) one easily sees that

$$\left\| \binom{\phi_{|k|,0} e^{ik\theta}}{0} \right\|_{\tilde{H}}^2 = - \int_R \phi_{|k|,0} e^{ik\theta} \overline{\Delta(\phi_{|k|,0} e^{ik\theta})} dxdy$$

$$+ \int_{\partial R} \phi_{|k|,0} e^{ik\theta} \overline{\frac{\partial \phi_{|k|,0}}{\partial r}} e^{-ik\theta} d\theta + \frac{|k| + 1}{\pi} \int_0^{2\pi} |k| d\theta$$

$$= 2|k|(|k| + 1), \quad -\infty < k < \infty , \tag{6.19}$$

while

$$\left\| \begin{pmatrix} \phi_{|k|,\ell} \\ \pm i\omega_{|k|,\ell}\phi_{|k|,\ell} \end{pmatrix} \right\|_{\tilde{H}}^2 = \lambda_{|k|,\ell} \int_R |\varepsilon_{|k|,\ell}|^2 dxdy$$

$$+ \int_R \nabla\phi_{|k|,\ell} \cdot \overline{\nabla\phi_{|k|,\ell}}\, dxdy = 2\lambda_{|k|,\ell} \int_R |\phi_{|k|,\ell}|^2 dxdy = 2\lambda_{|k|,\ell} \tag{6.20}$$

where

$$\lambda_{|k|,\ell} = (\omega_{|k|,\ell})^2, \quad -\infty < k < \infty, \quad \ell = 1,2,3,\dots .$$

The state $\binom{\phi_{0,0}}{0}$ has zero norm in $\tilde{H}$. Nevertheless we will not neglect this component.

If v, $\tilde{v}$ both satisfy the wave equation and (6.3), (4.13) on ∂R with initial state (4.5) for v we have (cf. (4.16))

$$\left(\binom{v(\cdot,\cdot,T)}{z(\cdot,\cdot,T)}, \binom{\tilde{v}(\cdot,\cdot,T)}{\tilde{z}(\cdot,\cdot,T)} \right)_{\tilde{H}} = \alpha \int_0^T \int_{B=\partial R} u(x,y,t)\, \overline{\frac{\partial\tilde{v}}{\partial\nu}(x,y,t)}\, dsdt . \tag{6.21}$$

It may be shown that this result is valid for all u for which the solution (in the generalized sense) v lies in $\tilde{H}$ and varies continuously with respect to t. This class of controls u is discussed in [19] and is known to include, e.g., $u \in C([0,T];H^{1/2}(B))$. If we assume $\binom{v}{z}$ given by the $\tilde{H}$- convergent series

$$\binom{v(\cdot,\cdot,t)}{z(\cdot,\cdot,t)} = \sum_{k=-\infty}^{\infty} v_{k,0}(t)(\phi_{|k|,0}e^{ik\theta}) +$$

$$\sum_{k=-\infty}^{\infty} \sum_{\ell=1}^{\infty} \left[v_{k,\ell}^{+}(t) \begin{pmatrix} \phi_{|k|,\ell}e^{ik\theta} \\ i\omega_{|k|,\ell}\phi_{|k|,\ell}e^{ik\theta} \end{pmatrix} + v_{k,\ell}^{-}(t) \begin{pmatrix} \phi_{|k|,\ell}e^{ik\theta} \\ -i\omega_{|k|,\ell}\phi_{|k|,\ell}e^{ik\theta} \end{pmatrix} \right]$$

and successively let

$$\begin{pmatrix} \tilde{v}(\cdot,\cdot,t) \\ \tilde{z}(\cdot,\cdot,t) \end{pmatrix} = \begin{pmatrix} \phi_{|k|,0}e^{ik\theta} \\ 0 \end{pmatrix}, \; e^{i\omega_{|k|,\ell}(t-T)} \begin{pmatrix} \phi_{|k|,\ell}e^{ik\theta} \\ i\omega_{|k|,\ell}\phi_{|k|,\ell}e^{ik\theta} \end{pmatrix},$$

$$e^{-i\omega_{|k|,\ell}(t-T)} \begin{pmatrix} \phi_{|k|,\ell}e^{ik\theta} \\ -i\omega_{|k|,\ell}\phi_{|k|,\ell}e^{ik\theta} \end{pmatrix}, \quad \begin{matrix} -\infty < k < \infty \\ \ell = 1,2,3,\dots \end{matrix} \tag{6.22}$$

for $T > 0$ we arrive at the equations

$$2|k|(|k|+1)v_{k,0}(T) = \alpha \int_0^T \int_0^{2\pi} u(\theta,t) \overline{\frac{\partial\phi_{|k|,0}}{\partial r}(1)}\, e^{-ik\theta} d\theta dt$$

$$= 2\pi\alpha \overline{\frac{\partial\phi_{|k|,0}}{\partial r}(1)} \int_0^T u_k(t)dt \,, \tag{6.23}$$

$$2\lambda_{|k|,\ell} v^+_{k,\ell}(T) = \alpha \int_0^T \int_0^{2\pi} u(\theta,t) e^{i\omega_{|k|,\ell}(T-t)} \overline{\frac{\partial\phi_{|k|,\ell}}{\partial r}(1)}\, e^{-ik\theta} d\theta dt$$

$$= 2\pi\alpha \overline{\frac{\partial\phi_{|k|,\ell}}{\partial r}(1)} \int_0^T e^{i\omega_{|k|,\ell}(T-t)} u_k(t)dt \,, \tag{6.24}$$

$$2\lambda_{|k|,\ell} v^-_{k,\ell}(T) = \alpha \int_0^T \int_0^{2\pi} u(\theta,t) e^{-i\omega_{|k|,\ell}(T-t)} \overline{\frac{\partial\phi_{|k|,\ell}}{\partial r}(1)}\, e^{-ik\theta} d\theta dt$$

$$= 2\pi\alpha \overline{\frac{\partial\phi_{|k|,\ell}}{\partial r}(1)} \int_0^T e^{-i\omega_{|k|,\ell}(T-t)} u_k(t)dt \,. \tag{6.25}$$

Thus the Dirichlet boundary control problem for (6.8), (6.10) is reduced to a moment problem (6.23), (6.24), (6.25) for which $u_k(t)$ must be a solution. We proceed in much the same way with the Neumann boundary control problem for (6.9), (6.11). We let

$$\zeta(r,\theta,t) = \sum_{k=-\infty}^{\infty} \zeta_k(r,t)e^{ik\theta} = \sum_{k=-\infty}^{\infty} \frac{\partial w_k(r,t)}{\partial t} e^{ik\theta} = \frac{\partial w}{\partial t}(r,\theta,t)$$

and obtain, in place of (6.12),

$$\frac{\partial}{\partial t}\begin{pmatrix} w_k(r,t) \\ \zeta_k(r,t) \end{pmatrix} = \begin{pmatrix} 0 & I \\ M_{|k|} & 0 \end{pmatrix}\begin{pmatrix} w_k(r,t) \\ \zeta_k(r,t) \end{pmatrix} = M_{|k|}\begin{pmatrix} v_k(r,t) \\ z_k(r,t) \end{pmatrix} . \tag{6.26}$$

The boundary conditions are now

$$\frac{\partial w_k}{\partial r}(1,t) = \beta u_k(t), \qquad -\infty < k < \infty .$$

The eigenvalues of $M_{|k|}$ with the corresponding homogeneous boundary condition

$$\frac{\partial w_k}{\partial r}(1,t) = 0$$

are, for $k = 0$,

$$0, \quad \pm i\nu_{0,\ell}, \qquad \ell = 1,2,3,\ldots ,$$

where $\nu_{0,\ell}$ is the ℓ-th zero of the differentiated Bessel function, $J_0'(r)$, of order 0, and, for $k \neq 0$,

$$\pm i\nu_{|k|,\ell}, \qquad \ell = 1,2,3,\ldots ,$$

where $\nu_{k,\ell}$ is the ℓ-th zero of $J_k'(r)$. In the case $k = 0$ the eigenvalue 0 has double multiplicity. The special solutions taking the place of (6.22) in this case are

$$\begin{pmatrix} \tilde{w}(\cdot,\cdot,t) \\ \tilde{\zeta}(\cdot,\cdot,t) \end{pmatrix} = \begin{pmatrix} \psi_{00} \\ 0 \end{pmatrix}, \quad \begin{pmatrix} (t - T)\psi_{00} \\ \psi_{00} \end{pmatrix} \tag{6.27}$$

where ψ_{00} is such that (cf. (6.16))

$$\int_0^1 r\psi_{00}^2 \, dr = \frac{1}{2\pi}, \quad \text{i.e. } \psi_{00} = \frac{1}{\sqrt{\pi}} .$$

In all of the other cases the vector eigenfunctions take the form

$$\begin{pmatrix} \psi_{|k|,\ell}(r) \\ \pm i\nu_{|k|,\ell}\psi_{|k|,\ell}(r) \end{pmatrix}, \quad -\infty < k < \infty, \quad \ell = 1,2,3,\ldots$$

where

$$\psi_{|k|,\ell}(r) = B_{|k|,\ell} J_{|k|}(\nu_{|k|,\ell} r), \quad \begin{matrix} -\infty < k < \infty , \\ \ell = 1,2,3,\ldots , \end{matrix}$$

the normalization coefficients

$$B_{|k|,\ell} = \frac{\nu_{|k|,\ell}}{\sqrt{\pi}\,(\mu_{|k|,\ell} - k^2)^{1/2} J_{|k|}(\nu_{|k|,\ell})} \tag{6.28}$$

selected so that

$$\int_0^R r|\psi_{|k|,\ell}(r)|^2 dr = \frac{1}{2\pi} .$$

The corresponding special solutions of the homogeneous equation are

$$\begin{pmatrix} \tilde{w}(\cdot,\cdot,t) \\ \tilde{\zeta}(\cdot,\cdot,t) \end{pmatrix} = e^{i\nu_{|k|,\ell}(t-T)} \begin{pmatrix} \psi_{|k|,\ell} e^{ik\theta} \\ i\nu_{|k|,\ell}\psi_{|k|,\ell} e^{ik\theta} \end{pmatrix} ,$$

$$e^{-i\nu_{|k|,\ell}(t-T)} \begin{pmatrix} \psi_{|k|,\ell} e^{ik\theta} \\ -i\nu_{|k|,\ell}\psi_{|k|,\ell} e^{ik\theta} \end{pmatrix} . \tag{6.29}$$

As in (6.20) it may be seen that

$$\left\| \begin{pmatrix} \psi_{|k|,\ell} \\ \pm i\nu_{|k|,\ell}\psi_{|k|,\ell} \end{pmatrix} \right\|_{\tilde{H}}^2 = 2\mu_{|k|,\ell} , \quad \mu_{|k|,\ell} = (\nu_{|k|,\ell})^2 .$$

Let w satisfy the wave equation and (6.4) with $w(x,y,0) \equiv 0$, $\zeta(x,y,0) = \frac{\partial w}{\partial t}(x,y,0) \equiv 0$ in R. We expand $\binom{w}{\zeta}$ in the form

$$\begin{pmatrix} w(\cdot,\cdot,t) \\ \zeta(\cdot,\cdot,t) \end{pmatrix} = w_{00}(t) \begin{pmatrix} \psi_{00} \\ 0 \end{pmatrix} + \zeta_{00}(t) \begin{pmatrix} 0 \\ \psi_{00} \end{pmatrix}$$

$$+ \sum_{k=-\infty}^{\infty} \sum_{\ell=1}^{\infty} \left[w_{k,\ell}^{+}(t) \begin{pmatrix} \psi_{|k|,\ell} e^{ik\theta} \\ i\nu_{|k|,\ell}\psi_{|k|,\ell} e^{ik\theta} \end{pmatrix} + w_{k,\ell}^{-}(t) \begin{pmatrix} \psi_{|k|,\ell} e^{ik\theta} \\ -i\nu_{|k|,\ell}\psi_{|k|,\ell} e^{ik\theta} \end{pmatrix} \right]$$

If $\tilde{w}$ satisfies the wave equation and the homogeneous boundary condition (cf. (4.14))

$$\frac{\partial \tilde{w}}{\partial \nu}(x,y,t) = 0, \quad (x,y) \in B, \quad t > 0 ,$$

we find (cf. (4.16), (6.21)) that

$$\left(\begin{pmatrix} w(\cdot,\cdot,T) \\ \zeta(\cdot,\cdot,T) \end{pmatrix}, \begin{pmatrix} \tilde{w}(\cdot,\cdot,T) \\ \tilde{\zeta}(\cdot,\cdot,T) \end{pmatrix} \right)_{\tilde{H}} = \beta \int_0^T \int_{B=\partial R} u(x,y,t) \overline{\frac{\partial \tilde{w}}{\partial t}(x,y,t)} \, ds dt . \tag{6.30}$$

Employing (6.29), (6.3) successively for $\binom{\tilde{w}}{\tilde{\zeta}}$ we arrive at the equations, for $-\infty < k < \infty$, $\ell = 1,2,3,\ldots,$

$$2\mu_{|k|,\ell}w^{+}_{k,\ell}(T) = \beta \int_0^T \int_0^{2\pi} u(\theta,t) i\nu_{|k|,\ell} e^{i\nu_{|k|,\ell}(T-t)} \psi_{|k|,\ell}(1) e^{-ik\theta} d\theta dt$$

$$\beta i\nu_{|k|,\ell}\overline{\psi_{|k|,\ell}(1)} \int_0^T e^{i\nu_{|k|,\ell}(T-t)} u_k(t)dt \ , \tag{6.31}$$

$$2\mu_{|k|,\ell}w^{-}_{k,\ell}(T) = -\beta \int_0^T \int_0^{2\pi} u(\theta,t) i\nu_{|k|,\ell} e^{-i\nu_{|k|,\ell}(T-t)} \psi_{|k|,\ell}(1) e^{-ik\theta} d\theta dt$$

$$= -2\pi\beta i\nu_{|k|,\ell}\overline{\psi_{|k|,\ell}(1)} \int_0^T e^{-i\nu_{|k|,\ell}(T-t)} u_k(t)dt \ . \tag{6.32}$$

We find also, taking $\begin{pmatrix}\tilde{w}\\ \tilde{\zeta}\end{pmatrix}$ in the second form given in (6.27), that

$$\zeta_{00}(T) = \beta \int_0^T \int_0^{2\pi} u(\theta,t)\overline{\psi_{00}} d\theta dt = 2\pi\beta\overline{\psi_{00}} \int_0^T u_0(t)dt \ . \tag{6.33}$$

Since this must be true for all T and $\frac{d}{dt} w_{00}(t) = \zeta_{00}(t)$, we have also

$$w_{00}(T) = 2\pi\beta\overline{\psi_{00}} \int_0^T (T-t)u_0(t)dt \tag{6.34}$$

Since $\mu_{|k|,\ell} = (\nu_{|k|,\ell})^2$, (6.31), (6.32) become

$$\frac{\nu_{|k|,\ell}}{\pi\beta i} w^{+}_{k,\ell}(T) = \overline{\psi_{|k|,\ell}(1)} \int_0^T e^{i\nu_{|k|,\ell}(T-t)} u_k(t)dt$$

$$= B_{|k|,\ell} J_{|k|,\ell}(\nu_{|k|,\ell}) \int_0^T e^{i\nu_{|k|,\ell}(T-t)} u_k(t)dt \tag{6.35}$$

$$\frac{\nu_{|k|,\ell}}{\pi\beta i} w^{-}_{k,\ell}(T) = -B_{|k|,\ell} J_{|k|,\ell}(\nu_{|k|,\ell}) \int_0^T e^{-i\nu_{|k|,\ell}(T-t)} u_k(t)dt \ . \tag{6.36}$$

Taking account of the fact that

$$\overline{\frac{\partial\phi_{|k|,\ell}}{\partial r}(1)} = \omega_{|k|,\ell} A_{|k|,\ell} \frac{\partial J_{|k|,\ell}}{\partial r}(\omega_{|k|,\ell})$$

(6.24) and (6.25) yield

$$\frac{\omega_{|k|,\ell}}{\pi\alpha} v^{+}_{k,\ell}(T) = A_{|k|,\ell} \frac{\partial J_{|k|,\ell}}{\partial r} (\omega_{|k|,\ell}) \int_0^T e^{i\omega_{|k|,\ell}(T-t)} u_k(t)dt \ , \tag{6.37}$$

$$\frac{\omega_{|k|,\ell}}{\pi\alpha} v^{-}_{k,\ell}(T) = A_{|k|,\ell} \frac{\partial J_{|k|,\ell}}{\partial r} (\omega_{|k|,\ell}) \int_0^T e^{-i\omega_{|k|,\ell}(T-t)} u_k(t)dt \ . \tag{6.38}$$

On the other hand

$$\frac{\partial \phi_{|k|,0}}{\partial r}(1) = A_{|k|,0}|k|$$

so (6.23) gives

$$\frac{|k| + 1}{\pi\alpha} v_{k,0}(T) = A_{|k|,0} \int_0^T u_k(t)dt \ . \tag{6.39}$$

Using the formula (6.18) and (6.28) for $A_{|k|,\ell}$ and $B_{|k|,\ell}$ we have

$$\frac{\nu_{|k|,\ell}}{\pi\beta i} w^{+}_{k,\ell}(T) = \frac{\nu_{|k|,\ell}}{\sqrt{\pi}\ (\mu_{|k|,\ell} - k^2)^{1/2}} \int_0^T e^{i\nu_{|k|,\ell}(T-t)} u_k(t)dt \tag{6.40}$$

$$\frac{\nu_{|k|,\ell}}{\pi\beta i} w^{-}_{k,\ell}(T) = \frac{-\nu_{|k|,\ell}}{\sqrt{\pi}\ (\mu_{|k|,\ell} - k^2)^{1/2}} \int_0^T e^{-i\nu_{|k|,\ell}(T-t)} u_k(t)dt \tag{6.41}$$

$$\frac{\omega_{|k|,\ell}}{\pi\alpha} v^{+}_{k,\ell}(T) = \frac{1}{\sqrt{\pi}} \int_0^T e^{i\omega_{|k|,\ell}(T-t)} u_k(t)dt \tag{6.42}$$

$$\frac{\omega_{|k|,\ell}}{\pi\alpha} v^{-}_{k,\ell}(T) = \frac{1}{\sqrt{\pi}} \int_0^T e^{-i\omega_{|k|,\ell}(T-t)} u_k(t)dt \tag{6.43}$$

The equations (6.39) become, in view of (6.17),

$$\frac{\sqrt{2}\ \sqrt{|k|(|k| + 1)}}{\pi\alpha} v_{k,0}(T) = \frac{\sqrt{2|k|}}{\sqrt{\pi}} \int_0^T u_k(t)dt \ . \tag{6.44}$$

This is valid, but meaningless, for $k = 0$. It is easy to see that in the case $k = 0$ we should use

$$\frac{1}{\sqrt{\pi}\ \alpha} v_{00}(T) = \int_0^T u_k(t)dt \ . \tag{6.45}$$

The equations (6.33) and (6.34) are left as they appear. We note that all of the coefficients

$$\frac{\nu_{|k|,\ell}}{\sqrt{\pi}\ (\mu_{|k|,\ell} - k^2)^{1/2}},\quad \frac{1}{\sqrt{\pi}},\quad \frac{\sqrt{2|k|}}{\sqrt{\pi}},\quad k \neq 0,\quad 2\pi\beta \tag{6.45}$$

are bounded away from zero, uniformly with respect to k.

It is also possible to show, using the work [10], [11] of K. D. Graham, that the numbers

$$0,\ \nu_{|k|,1},\ \omega_{|k|,1},\ \nu_{|k|,2},\ \omega_{|k|,2},\dots,\nu_{|k|,j},\ \omega_{|k|,j},\dots$$

are separated by a gap at least equal to $\pi/2$ again uniformly with respect to k. Applying the result [14] of A. E. Ingham along with the work of Duffin and Schaeffer [7], much as in [12], [2], [3], we conclude the existence of functions $u_k(t)$ in $L^2[0,T]$, for *any fixed* $T > 4$, *solving the above moment problems*, $-\infty < k < \infty$. Moreover, the result of Ingham implies as explained in [12], [26], that for each k

$$C^{-2}N_k^2 \le \int_0^T |u_k(t)|^2 dt \le C^2 N_k^2$$

where

$$N_k^2 = 2|k|(|k| + 1)|v_{k,0}(T)|^2$$
$$+ \sum_{\ell=1}^{\infty} \lambda_{|k|,\ell}|v_{k,\ell}^{+}(T)|^2 + \sum_{\ell=1}^{\infty} \lambda_{|k|,\ell}|v_{k,\ell}^{-}(T)|^2$$
$$+ \sum_{\ell=1}^{\infty} \mu_{|k|,\ell}|w_{k,\ell}^{+}(T)|^2 + \sum_{\ell=1}^{\infty} \mu_{|k|,\ell}|w_{k,\ell}^{-}(T)|^2$$

$k = \pm 1, \pm 2, \dots$. For $k = 0$ we must add $|\zeta_{00}(T)|^2 + |w_{00}(T)|^2$. Since

$$\int_0^T \int_0^{2\pi} |u(\theta,t)|^2 d\theta dt = \sum_{k=-\infty}^{\infty} \int_0^T |u_k(t)|^2 dt \tag{6.46}$$

we see that the above moment problems, equivalent to the control problem, can be solved with (6.46) finite, provided that

$$\sum_{k=-\infty}^{\infty} N_k^2 < \infty ,$$

which is the same as saying that the norm of the final state in $\hat{H}$ should be finite. We have, then, the exact controllability result that any $\hat{H}$ state may be controlled to any other $\hat{H}$ state during a time interval of length $T > 4$ with the control configuration we have described here. As discussed in connection with the wave equation in [FF], [GG], one cannot be sure that the state of the system remains in $\hat{H}$ for all $t \in [0,T]$. However, in the present case of the Maxwell equations one can show that these states do lie in $H = H_{\mathbf{E},d}(\mathbf{R})$.

7. CONCLUDING REMARKS

The approximate controllability results of Section 5 would appear to be extendable to domains other than rectangular ones but the precise method of extension remains to be worked out. We will indicate some aspects of this problem which are clear from our current work.

First of all, the result of Section 5 is almost trivially extended to the case where control is exercised only on a subset $\{(\pi,y) | 0 < a < y < b < \pi\}$, $b > a$, of $\{(\pi,y) | 0 < y < \pi\}$. The only change is that the interval $1 + \delta < t < T - 1 - \delta$ appearing in (5.14) and subsequently must be modified to $d + \delta < t < T - d - \delta$ where

$$d = \inf_{a<y<b} \left\{ \sup_{\substack{0<\xi<\pi \\ 0<\eta<\pi}} \{[(\pi - \xi)^2 + (\eta - y)^2]^{1/2}\}\right\} .$$

If $\phi(\pi,y,t) \equiv \frac{\partial \phi}{\partial x}(\pi,y,t) \equiv 0$ for $\delta < t < T - \delta$, $a < y < b$, the Holmgren theorem will still apply to show that $\phi(x,y,t) \equiv 0$, $(x,y) \in R$, $d + \delta < t < T - d - \delta$. After that the remainder of the proof is the same: the same eigenfunctions and frequencies must be dealt with, the functions $\sin jy$, $\cos jy$ are still independent on $a < y < b$ if $b > a$ and the conditions

$$\hat{v}_0(x,y) = 0, \quad (x,y) \in B - \{(\pi,y) | a < y < b\}$$

$$\frac{\partial \hat{v}_0}{\partial x}(\pi,y) = 0, \quad a < y < b ,$$

still show $\hat{v}_0(x,y) \equiv 0$ in R.

The first limitation of the method which we have used in Section 5 lies in its dependence on the construction of $\phi(x,y,t)$ as a linear combination of partial derivatives of $\hat{v}$ and $\hat{w}$. It is necessary to have a solution of the wave equation to which Holmgren's theorem may be applied. This part of the proof can still be used for non-rectangular domains as long as a portion of the boundary on which control is applied is a straight line segment. Assuming the segment parallel to the y-axis, one can construct ϕ by the formula (5.8) again and show that ϕ and $\frac{\partial \phi}{\partial x}$ both vanish on the straight line segment in question, allowing subsequent application of the Holmgren theorem to show $\phi(x,y,t) \equiv 0$

for $(x,y) \in R$ and t in some interval $d + \delta < t < T - d - \delta$, with d depending on the geometry of R. But then we are faced with a second limitation.

The second limitation of the method which we have used lies in its reliance on the specific form of the eigenfunctions and frequencies to pass from $\phi(x,y,t) \equiv 0$ to the conclusion that both $\hat{v}(x,y,t)$ and $\hat{w}(x,y,t)$ are likewise identically zero. It needs to be emphasized that no local analysis will suffice here. In the one dimensional case (see our remarks at the end of Section 3) if the control problem is stated for boundary conditions

$$v(0,t) = 0, \quad \frac{\partial v}{\partial t}(1,t) = \alpha u(t) \tag{7.1}$$

$$\frac{\partial w}{\partial x}(0,t) = 0, \quad \frac{\partial w}{\partial x}(1,t) = \beta u(t) \tag{7.2}$$

the $\tilde{v}$, $\tilde{w}$ constructed as in Section 4 will satisfy the wave equation and

$$\tilde{v}(0,t) = 0, \quad \frac{\partial \tilde{v}}{\partial t}(1,t) = 0\ , \tag{7.3}$$

$$\frac{\partial \tilde{w}}{\partial x}(0,t) = 0, \quad \frac{\partial \tilde{w}}{\partial x}(1,t) = 0\ , \tag{7.4}$$

$$\alpha \frac{\partial \tilde{v}}{\partial x}(1,t) + \beta \frac{\partial \tilde{w}}{\partial t}(1,t) \equiv \phi(1,t) = 0 \tag{7.5}$$

Here if we take $\tilde{w}$ to be a non-zero solution of the wave equation satisfying (7.4) and take

$$\tilde{v}(x,t) = -\frac{\beta}{\alpha}\int_0^x \frac{\partial \tilde{w}}{\partial t}(\xi,t)d\xi$$

we clearly have $\tilde{v}(0,t) = 0$,

$$\frac{\partial \tilde{v}}{\partial t}(1,t) = \frac{\beta}{\alpha}\int_0^1 \frac{\partial^2 \tilde{w}}{\partial t^2}(\xi,t)d\xi$$

$$= -\frac{\beta}{\alpha}\int_0^1 \frac{\partial^2 \tilde{w}}{\partial \xi^2}(\xi,t)d\xi = \frac{\beta}{\alpha}\left(\frac{\partial \tilde{w}}{\partial x}(0,t) - \frac{\partial \tilde{w}}{\partial x}(1,t)\right) = 0\ ,$$

$$\frac{\partial^2 \tilde{v}}{\partial t^2}(x,t) = -\frac{\beta}{\alpha}\int_0^x \frac{\partial^3 \tilde{w}}{\partial t^3}(\xi,t)d\xi$$

$$= -\frac{\beta}{\alpha}\int_0^x \frac{\partial^3 \tilde{w}}{\partial t \partial \xi^2}(\xi,t)d\xi = -\frac{\beta}{\alpha}\frac{\partial^2 \tilde{w}}{\partial t \partial x}(x,t) = \frac{\partial^2 \tilde{v}}{\partial x^2}(x,t)$$

so that $\tilde{v}$ satisfies the wave equation and, clearly, (7.5) is also satisfied. Thus the wave equation with (7.1), (7.2) is not approximately controllable; $\phi(x,t) \equiv \alpha \frac{\partial \tilde{v}}{\partial x}(x,t) + \beta \frac{\partial \tilde{w}}{\partial t}(x,t) \equiv 0$ but this does not imply that $\tilde{v}$ or $\tilde{w}$ are identically equal to zero. The additional condition which makes this work in (3.7) ff. is the fact that one can show there that

$$-\alpha \frac{\partial \tilde{v}}{\partial x}(0,t) + \beta \frac{\partial \tilde{w}}{\partial t}(0,t) = 0 .$$

It seems likely that the question of whether or not $\phi = 0$ implies that both $\hat{v}$ and $\hat{w}$, equivalently $\tilde{v}$ and $\tilde{w}$, are both zero must eventually reduce to a boundary value problem of an as yet unidentified type.

At the present writing there is only one, rather curious, result which we can offer which yields approximate controllability for a domain R of rather general shape. We suppose that the "control boundary" B_1 $B = \partial R$ includes two nonparallel line segments, ℓ_1 and ℓ_2, with unit exterior normals ν_1 and ν_2. Proceeding as before we can show, applying the Holmgren theorem together with

$$\frac{\partial \hat{v}}{\partial t} = 0 \quad \text{on} \quad \ell_1, \ell_2$$

$$\frac{\partial \hat{w}}{\partial \nu_i} = 0, \quad i = 1,2 \quad \text{on} \quad \ell_1, \ell_2, \quad \text{respectively,}$$

$$\alpha \frac{\partial \hat{v}}{\partial \nu_i} + \beta \frac{\partial \hat{w}}{\partial t} = 0, \quad i = 1,2 \quad \text{on} \quad \ell_1, \ell_2 \quad \text{respectively,}$$

that both

$$\phi_1 = \alpha \frac{\partial \hat{v}}{\partial \nu_1} + \beta \frac{\partial \hat{w}}{\partial t}, \tag{7.6}$$

$$\phi_2 = \alpha \frac{\partial \hat{v}}{\partial \nu_2} + \beta \frac{\partial \hat{w}}{\partial t}, \tag{7.7}$$

must vanish identically in R for $d + \delta \le t \le T - d - \delta$, $\delta > 0$ arbitrary, $d > 0$ depending on the geometry of R and B, the location of ℓ_1 and ℓ_2 within B, etc. But then both ϕ_1 and ϕ_2 must vanish on ℓ_1 (say) for these values of t. Subtracting (7.6) from (7.7) we see that

$$\alpha\left(\frac{\partial \hat{v}}{\partial \nu_1} - \frac{\partial \hat{v}}{\partial \nu_2}\right) = 0 \quad \text{on} \quad \ell_1 \times [d + \delta, T - d - \delta]$$

This shows, since ℓ_1 and ℓ_2 are not parallel, that a nontangential derivative of $\hat{v}$ vanishes on $\ell_1 \times [d + \delta, T - d - \delta]$. Combining this with $\frac{\partial \hat{v}}{\partial t} = 0$ on ℓ_1 and applying the Holmgren theorem to $\hat{v}$ alone, much as in [5], [13], we are able to conclude $\hat{v} \equiv 0$, provided T is appropriately large. Then one easily has the same result for $\hat{w}$ and approximate controllability follows.

This result gives approximate controllability for R equal to the interior of any closed polyhedron in R^2 with control on at least two sides.

Further inspection of this argument shows that only ℓ_2 needs to be assumed to be a line segment. That is needed in order to identify ϕ_2 as a solution of the wave equation. We may then take ℓ_1 to be any smooth portion of B_1 which is never parallel to ℓ_1 and achieve the same result.

Finally, let us indicate that we are very much aware of the limitations, from the point of view of actual implementation, of the control configuration discussed in this paper. In principle, at least, the boundary conditions (1.7), (1.8), along with the further "single layer" condition discussed in connection with Figure 3.1, could be achieved with conducting bars attached to terminals as shown in Figure 3.

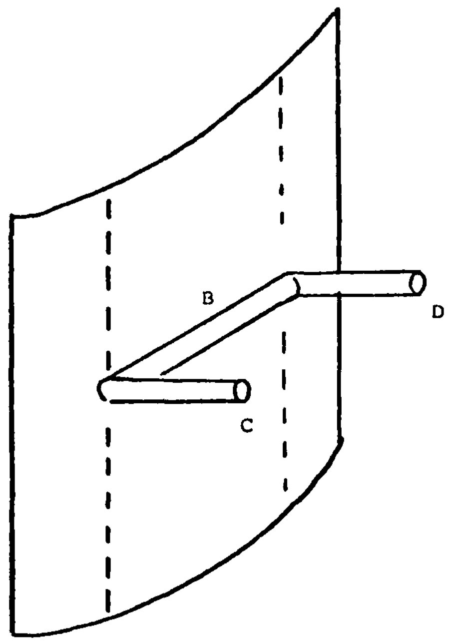

Figure 3. Conducting Bar and Busses

The perfectly conducting busses perpendicular to the boundary of Ω ensure that the normal component of $\vec{E}$, E_ν, is zero just outside Ω, provided that no net change is allowed to accumulate at the boundary of Ω, i.e., in the conducting bar. Thus the potentials at C and D must be regulated so that the potential difference C − D ensures the correct controlling current through the surface bar B while C + D is set so that there is no accumulation of charge at the bounding surface.

We have not considered any effects of propagation delays in the controlling circuits – i.e., we have not assumed that these are distributed parameter systems. This assumption, and evident limitations on the speed with which prescribed currents can be computed and established in the controlling circuits together with sensing limitations, place admittedly severe limitations on what can be done "open loop". It is likely that the eventual significance of our results will be most evident in connection with closed loop behavior wherein time varying magnetic fields $\vec{H}$ near the boundary of Ω induce currents in the bars B which, being resistive, will then act as energy dissipators. We hope to discuss this topic in later work.

Another control configuration may be obtained by supposing the boundary of Ω to be a perfectly conducting sheet of material to which electromagnets are attached in a dense array as shown in Figure 4.

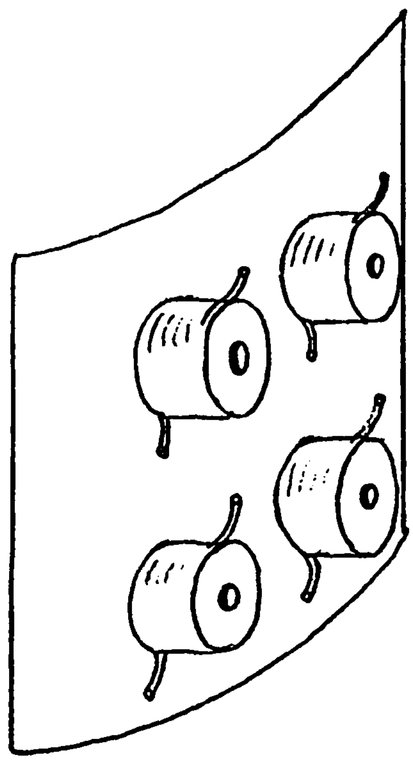

Figure 4. Electromagnet Array

If J denotes the current through the windings of the electromagnets, then we shall have

$$\vec{E}_\tau = 0$$

and

$$H_\nu = \alpha J$$

where α is dependent on the electromagnet's configuration. The theory in this case will take much the same form as the one discussed in this paper.

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Dual Paley-Wiener Spaces and

"Regular" Nonharmonic Fourier Series*

by

David L. Russell**

Abstract

We present here a class of realizations $\{\Psi_p\}$ of the dual space Φ' for the Paley-Wiener (Hilbert) space Φ of entire functions. The elements of each space Ψ_p are meromorphic functions with poles at the zeros, z_k, $k \in K$, of a certain "cardinal function" p. The relationships between Φ and Ψ_p are explored and applications are made to the study of nonharmonic Fourier series whose terms are complex exponentials $e^{z_k t}$.

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APPENDIX II

Dual Paley-Wiener Spaces and "Regular" Nonharmonic Fourier Series

1. Introduction and Statement of Principal Results.

It is well known that certain families of entire functions may be given a Hilbert space structure. (See, in particular, the extensive work [A] of de Branges in this connection.) The most familiar of these spaces is the so-called "Paley-Wiener space", which we here designate as Φ. It consists of entire functions $\phi(z) = \phi(\xi+i\eta)$ with the following properties: For each $\phi \in \Phi$

(i) there exists a positive number, M_ϕ, such that

$$|\phi(x)| < M_\phi e^{\pi|\xi|}, \qquad z = \xi+i\eta \in \mathbb{C}; \tag{1.1}$$

(ii) there exists a positive number, N_ϕ, such that for every real ξ

$$\int_{-\infty}^{\infty} |\phi(\xi+i\eta)|^2 d\eta < N_\phi e^{2\pi|\xi|}. \tag{1.2}$$

An inner product and norm for this space are described in [A] and that norm is equivalent to the norms which we will introduce at the beginning of Section 2.

One of the purposes of this article is to introduce a space (actually, a class of spaces), Ψ, of analytic functions $\psi = \psi(z)$ having singularities confined to a vertical strip in the complex plane $\mathbb{C}$, and serving as a natural representation of Φ', the dual space to Φ. The main interest centers on $\psi \in \Psi$ which are meromorphic with poles confined to such a strip. The relationship between Φ and Ψ is somewhat similar to the duality relationship between paired H^2_α spaces. If we define the left and right Hardy spaces G^2_α and H_α to consist of functions $g(z)$, $h(z)$, analytic in $\mathrm{Re}(z) < \alpha$, $\mathrm{Re}(z) > \alpha$, respectively, bounded in sets $\mathrm{Re}(z) < \alpha-\varepsilon$, $\mathrm{Re}(z) > \alpha-\varepsilon$, respectively, and satisfying uniform L^2 bounds

$$\int_{-\infty}^{\infty} |g(\xi+i\eta)|^2 d\eta < B_g, \quad \xi < \alpha,$$

$$\int_{-\infty}^{\infty} |h(\xi+i\eta)|^2 d\eta < C_h, \quad \xi > \alpha,$$

then (see, e.g., [B]) g and h have L^2 traces on the line $\mathrm{Re}(z) = \xi = \alpha$ and the duality relationship

$$\langle g,h\rangle \equiv \int_{-\infty}^{\infty} g(\alpha+i\eta)h(\alpha+i\eta)d\eta$$

may be used to define all linear functionals on G_α^2 or H_α^2, each of these spaces being a natural representation of the dual space of the other. We will have more to say about this in Section 3.

Just as in the case of the Paley-Wiener space and the other, related, spaces described by de Branges, the spaces Ψ which we introduce as dual spaces to Φ are intimately connected with certain entire functions $p(z)$ which "just fail" to lie in Φ; p does not belong to Φ but if $\hat{z}$ is one of the zeros of p, $p(z)/(z-\hat{z})$ does belong to Φ. We call such a function a cardinal function. The precise definition of a cardinal function operative in this paper is the following: an entire function of order 1 and type π, $p(z)$, is a (regular) cardinal function if there exist M^+, M^-, α, all positive, such that, for all $z = \xi+i\eta$,

$$|p(\xi+i\eta)| < M^+ e^{\pi|\xi|} \tag{1.3}$$

and

$$|p(\xi+i\eta)| > M^- e^{\pi|\xi|}, \quad |\xi| > \alpha. \tag{1.4}$$

If p is a cardinal function, the space of meromorphic functions

$$\Psi_p = \{\psi \mid \psi(z) = \phi(z)/p(z),\ \phi \in \Phi\}$$

is shown to be a natural representation of the dual space Φ'. With Z_p being the set of zeros of p, one sees that the meromorphic functions $\psi \in \Psi_p$ have partial fraction decompositions analogous to

$$\psi(z) = \sum_{\zeta \varepsilon Z_p} \frac{c_\zeta}{z-\zeta}$$

(with suitable modifications in the case of multiple zeros) which are related, in much the same manner as described by Schwartz in [C], to exponential bases

$$E_p = \{e^{z_k t} \mid z_k \varepsilon Z_p\}$$

for the space $L^2[-\pi,\pi]$. We are able in this way, to describe certain Riesz bases and "uniform decompositions" of $L^2[-\pi,\pi]$, using properties of p somewhat different from the assumptions on the growth and spacing of its zeros appearing in the classical work of Paley and Wiener [D], Levinson [E] and Schwartz [C], or in more recent treatments, such as Duffin and Schaeffer [F] and Young [G].

2. Φ and Ψ as Spaces of Fourier and Laplace Transforms.

The linear vector space, Φ, of entire functions satisfying (1.1) and (1.2) coincides, as is well known, with the set of Fourier transforms

$$\phi(z) = \int_{-\pi}^{\pi} e^{zt} f(t)dt \equiv (\mathcal{F} f)(z) \tag{2.1}$$

corresponding to functions $f \in L^2[-\pi,\pi]$. The inverse relationship is

$$f(t) = \underset{A\to\infty}{\mathrm{l.i.m.}} \frac{1}{2\pi i} \int_{\xi-iA}^{\xi+iA} e^{-zt} \phi(z)dz \equiv (\mathcal{F}^{-1}\phi)(t), \tag{2.2}$$

the integration taking place over the straight line segment joining the two integration limits. The Plancherel formula

$$\|f\|^2_{L^2[-\pi,\pi]} = \frac{1}{2\pi} \|\phi(i\cdot)\|^2_{L^2(-\infty,\infty)}$$

shows that (2.1) and (2.2) are each positive scalar multiples of an isometry on $L^2(-\infty,\infty)$, the notation $\phi(i\cdot)$ indicating the restriction of ϕ to the imaginary axis. From

$$\phi(\xi+i\eta) = \int_{-\pi}^{\pi} e^{(\xi+i\eta)t} f(t)dt$$

it is easy to see that for each real ξ

$$\begin{aligned} e^{-2\pi|\xi|} \|\phi(i\cdot)\|^2_{L^2(-\infty,\infty)} &\leq \int_{-\infty}^{\infty} |\phi(\xi+i\eta)|^2 d\eta \\ &\leq e^{2\pi|\xi|} \|\phi(i\cdot)\|^2_{L^2(-\infty,\infty)} \end{aligned} \tag{2.3}$$

from which it follows that each of the norms $\|\ \|_\rho$ defined by

$$\begin{aligned} \|\phi\|^2_\rho &= \int_{-\infty}^{\infty}(|\phi(\rho+i\eta)|^2 + |\phi(-\rho+i\eta)|^2)d\eta \\ &= \int_{\Gamma_\rho} |\phi(z)|^2 |dz| = \|\phi\|^2_{L^2(\Gamma_\rho)}, \end{aligned}$$

Γ_ρ being the contour consisting of $\mathrm{Re}\ z = \rho$, oriented upwards, and $\mathrm{Re}\ z = -\rho$, oriented downwards, is equivalent to $\|\phi(i\cdot)\|_{L^2(-\infty,\infty)}$. Much of our work depends upon being able to vary at will the particular value of ρ being using for $\|\ \|_\rho$, secure in the knowledge that the resultant topology remains invariant.

Let Ψ denote a certain family of functions $\psi(z)$ analytic in $|\mathrm{Re}(z)| > a$ for some $a \geq 0$ which may depend on ψ. With Γ_ρ as already defined, $\rho > a$, we specify Ψ precisely as consisting of such functions ψ for which the identity

$$\psi(z) = \frac{1}{2\pi i} \int_{\Gamma_\rho} \frac{\psi(\xi)}{z-\zeta} d\zeta, \quad |\mathrm{Re}(z)| > \rho, \tag{2.4}$$

is satisfied, and, also for every $\rho \geq \beta > a$

$$\int_{\Gamma_\rho} |\psi(z)|^2 |dz| \leq N_\beta , \tag{2.5}$$

where $N\beta$ is a positive number depending on β. It is quite straightforward to see that a sufficient condition for a function ψ, satisfying the second condition, (2.5), to also satisfy the first condition, (2.4), is that $|\psi(z)|$ should be bounded in $|\mathrm{Re}(z)| \geq \rho$ for every $\rho > a$ and, again for every $\rho > a$,

$$\lim_{r \to \infty} \int_{C_{r,\rho}} \frac{\psi(z)dz}{\rho-z} = \lim_{r \to \infty} \int_{C_{r,-\rho}} \frac{\psi(z)dz}{-\rho-z} = 0,$$

where $C_{r,\rho}$, $C_{r,-\rho}$ are, respectively, the right anid left hand semicircles of radius r centered at the points $z = \rho$, $z = -\rho$, respectively.

<u>Proposition 1.1. Corresponding to each ψ (and associated a) in Ψ there is a unique function</u> $g \in L^2_\rho(-\infty,\infty)$, $\rho > a$, <u>where</u>

$$L^2_\rho(-\infty,\infty) = \{g \in L^2_{loc}(-\infty,\infty) \mid \int_{-\infty}^{\infty} e^{-2\rho|t|} |g(t)|^2 dt < \infty\}, \tag{2.6}$$

<u>and</u> $\psi = g$, <u>the "two-sided " Laplace transform of</u> g, <u>in the sense that</u>

$$\psi(z) = (\mathcal{L}g)(z) = \int_0^\infty e^{-zt} g(t)dt, \quad \mathrm{Re}(z) > a, \tag{2.7}$$

$$= -\int_{-\infty}^0 e^{-zt} g(t)dt, \quad \mathrm{Re}(z) < -a. \tag{2.8}$$

Moreover, for each $g \in L^2_\rho(-\infty,\infty)$, $\psi(z) = (\mathcal{L}g)(z) \in \Psi$.

Proof. This is quite standard, so we will be brief. Symmetry allows us to consider only the $t > 0$ part of (2.6) and the first identity (2.7). Given $\psi \in \Psi$, we define $g \in L^2_\rho[0,\infty)$ by use of the Laplace inversion formula on the line $\mathrm{Re}(z) = \rho$, $\rho > a$, and application of the Plancherel Theorem. On the other hand, if $g \in L^2_\rho[0,\infty)$ and we define $\psi(z) = (\mathcal{L}g)(z)$ by (2.7) for $\mathrm{Re}(z) > a$, application of the Plancherel Theorem again establishes (2.5), insofar as the portion $\Gamma^+_\rho = \{z \mid \mathrm{Re}(z) = \rho\}$ is concerned, for $\rho > a$. For $\rho > \beta > a$, application of (2.7) readily shows that

$$|\psi(z)| \le \frac{1}{\mathrm{Re}(z)-\beta} \|e^{-\beta t} g(t)\|_{L^2[0,\infty)}, \quad \mathrm{Re}(z) \ge \rho. \tag{2.9}$$

Let $\Gamma_{r,\rho}$ denote the positively oriented D-shaped contour consisting of $C_{r,\rho}$, as defined earlier, and $\{z \mid \mathrm{Re}(z) = \rho, |\mathrm{Im}(z)| \le r\}$. If $\mathrm{Re}(w) > \rho$, then for sufficiently large r

$$\psi(w) = \frac{1}{2\pi i} \int_{\Gamma_{r,\rho}} \frac{\psi(z)dz}{w-z}. \tag{2.10}$$

For $z = \rho + re^{i\theta}$, $-\pi/2 \le \theta \le \pi/2$,

$$\frac{1}{\mathrm{Re}(z) - \beta} = \frac{1}{\mathrm{Re}(z) - \rho + (\rho-\beta)} = \frac{1}{r\cos\theta + (\rho-\beta)}$$

is bounded and tends uniformly to zero as $r\to\infty$ each sector $-\pi/2 + \delta \le \theta \le \pi/2 - \delta$, $\delta > 0$. Using (+) together with the fact that $|w-z|^{-1} = \mathcal{O}(r^{-1})$ uniformly on $C_{r,\rho}$ as $r = |z-\rho|$ tends to ∞, the integral over $C_{r,\rho}$ is seen to vanish as $r\to\infty$ and (2.4) follows from (2.10), the convergence of the integral in (2.4) again guaranteed by the Plancherel Theorem. The proof for $\mathrm{Re}(w) < -a$ is almost word for word the same so we will regard the proposition as proved.

For $g \in L^2_\rho[0,\infty)$, $\rho > a$, the usual Laplace inversion formula shows that

$$g(t) = \underset{A\to\infty}{\mathrm{l.i.m.}} \frac{1}{2\pi i} \int_{\rho-iA}^{\rho+iA} e^{zt}\psi(z)dz.$$

For $t < 0$ a standard argument shows that the integral vanishes. For $g \in L^2(-\infty,0]$ we have, for $-\rho < -a$,

$$g(t) = \underset{A\to\infty}{\mathrm{l.i.m.}} \frac{1}{2\pi i} \int_{-\rho-iA}^{-\rho+iA} e^{zt}\psi(z)dz.$$

and the integral vanishes for $t > 0$. Thus, letting

$$\Gamma_{\rho,A} = \Gamma_\rho \cap \{z \mid |\mathrm{Im}(z)| < A\} \tag{2.11}$$

we may write

$$g(t) = \underset{A\to\infty}{\mathrm{l.i.m.}} \frac{1}{2\pi i} \int_{\Gamma_{\rho,A}} e^{zt}\psi(z)dz. \tag{2.12}$$

Let p be a cardinal function as defined in Section 1. We define Ψ_p to be the subspace of Ψ consisting of functions ψ such that

$$\phi(z) = p(z)\psi(z) \tag{2.13}$$

is an entire function and the identity

$$\phi(z) = \frac{1}{2\pi i} \int_{\Gamma_\rho} \frac{\phi(\zeta)}{\zeta - z}\, d\zeta$$

is valid for all z in the open strip interior to Γ_ρ, $\rho > \alpha$ as defined for p in (1.4).

The results which we present next concern the structure of Ψ_p as it relates to Φ and the cardinal function p.

<u>Theorem 2.2. Let</u> $\phi \in \Phi$, <u>let</u> p <u>be a cardinal function, and let</u> α <u>be as specified in</u> (1.4). <u>Define</u>

$$(P^{-1}\phi)(z) \equiv \phi(z)/p(z) \equiv \psi(z). \tag{2.15}$$

Then $\psi \in \Psi_p \subset \Psi$ and for every $\rho > \alpha$ there is a positive μ_ρ such that

$$\int_{\Gamma_\rho} |\psi(z)|^2 |dz| \le \mu_\rho^2 \|\phi\|_\rho^2 . \tag{2.16}$$

Proof. Let $\rho > \alpha$ and let $|\mathrm{Re}(z)| > \rho$. Looking at $\mathrm{Re}(z) > \rho$ first, we have

$$\psi(z) = \frac{1}{2\pi i} \int_{\Gamma_{r,\rho}} \frac{\psi(\zeta)}{z-\zeta} d\zeta,$$

where $\Gamma_{r,\rho}$ is defined as in the proof of Proposition 1.1. Let $\phi(z) = (\ f)(z)$ as in (2.1). Then with $z = \xi + i\eta$

$$|\phi(z)|^2 \le \int_{-\pi}^{\pi} e^{2\xi t}\, dt\, \|f\|^2_{L^2[-\pi,\pi]}$$

$$= \frac{1}{2\xi}\left(e^{2\pi\xi} - e^{-2\pi\xi}\right) \|f\|^2_{L^2[-\pi,\pi]}$$

so that

$$|\phi(z)| \le \tilde{M}_\phi\, e^{\pi|\xi|}/(1+|\xi|)^{1/2},$$

with M_ϕ depending only on ϕ, not z. Using this with property (1.4) of p and applying the Jordan lemma we see that

$$\lim_{r\to\infty} \int_{C_{r,\rho}} \frac{\psi(\zeta)}{z-\zeta} d\zeta = 0$$

so that

$$\psi(z) = \frac{1}{2\pi i} \int_{\Gamma_\rho^+} \frac{\psi(\zeta)}{\zeta - z} d\zeta .$$

For $\mathrm{Re}(z) > \rho > \alpha$ the corresponding integral over Γ_ρ, the left hand portion of Γ_ρ, oriented downward, vanishes and thus

$$\psi(z) = \frac{1}{2\pi i} \int_{\Gamma_\rho} \frac{\psi(\zeta)}{\zeta - z} d\zeta . \tag{2.17}$$

A similar argument shows that (2.17) also applies for $\mathrm{Re}(z) < -\rho < -\alpha$. Hence condition (2.4) for ψ to be a member of Ψ is satisfied. Condition (2.5) follows immediately from (1.2) and (1.4). Since $p(z)\psi(z) = \phi(z)$ is entire, there remains only the proof of (2.14) to show that $\psi \in \Psi_\rho$. Define $R_{\rho,A}$ to be the positively oriented rectangle with corners $\pm\rho\pm iA$. For z interior to $R_{\rho,A}$ we clearly have

$$\phi(z) = \frac{1}{2\pi i}\int_{R_{\rho,A}} \frac{\phi(\zeta)}{\zeta-z}\, d\zeta \, . \tag{2.18}$$

From the bound (1.1) we have

$$\left|\frac{\phi(\zeta)}{\zeta-z}\right| < \frac{M_\phi e^{\pi\rho}}{A-\mathrm{Im}(z)}, \quad \zeta = r+iA, \qquad -\rho < r < \rho$$

and a similar bound holds for $\zeta = r-iA$. Hence, letting $A\to\infty$, (2.18) Becomes

$$\phi(z) = \frac{1}{2\pi i}\int_{\Gamma_\rho} \frac{\psi(\zeta)}{\zeta-z}\, d\zeta \, , \tag{2.19}$$

which corresponds to (2.14). We conclude $\psi \in \Psi_p$ and the proof is complete.

The next theorem is a complementary result to Theorem 2.2. Its proof is only slightly more difficult.

<u>Theorem 2.3.</u> <u>Let</u> $\psi \in \Psi_p$ <u>and let</u>

$$\phi(z) = p(z)\psi(z) \equiv (P\psi)(z). \tag{2.20}$$

<u>Then</u> $\phi \in \Phi$.

<u>Proof.</u> Let a be as specified for ψ preceding (2.4). For $|\mathrm{Re}(z)| > \beta > \rho > a$, (2.4) and (2.5) combine, using the Schwartz inequality, to show that for some $B_\beta > 0$

$$|\psi(z)| < B_\beta.$$

Since this is true for every $\beta > a$, using (2.20) with property (1.3) of p we have

$$\left|\phi(z)\right| < M_{\beta} e^{\pi|\xi|}, \quad z = \xi + i\eta, \quad |\xi| > \beta > \hat{a} = \mathrm{Max}(a,\alpha), \tag{2.21}$$

α specified for p as in (1.4). Then using (2.5) with property (1.3) of p we have

$$\int_{\Gamma_{\tilde{\rho}}} \left|\phi(z)\right|^2 |dz| < N_{\beta}(M^+)^2 e^{2\pi\tilde{\rho}}, \quad \tilde{\rho} > \beta > \hat{a}\,. \tag{2.22}$$

The inequalities (2.21) and (2.22) establish (1.1) and (1.2) for $|\mathrm{Re}(z)| = |\xi| > \beta > \hat{a}$. There remains the question of the behavior of $\phi(z)$ inside a strip $|\mathrm{Re}(z)| < \tilde{\rho}$, $\tilde{\rho} > \beta$ in order to complete the proof.

Let the right and left halves of Γ_ρ, oriented upwards and downwards respectively, be denoted by $\Gamma^+_{\tilde{\rho}}$, $\Gamma^-_{\tilde{\rho}}$, respectively. Define

$$\phi^+(z) = \frac{1}{2\pi i}\int_{\Gamma^+_{\tilde{\rho}}} \frac{\phi(\zeta)}{\zeta - z}\,d\zeta, \qquad \mathrm{Re}(z) < \tilde{\rho}, \tag{2.23}$$

$$\phi^-(z) = \frac{1}{2\pi i}\int_{\Gamma^-_{\tilde{\rho}}} \frac{\phi(\zeta)}{\zeta - z}\,d\zeta, \qquad \mathrm{Re}(z) < -\tilde{\rho}\,. \tag{2.24}$$

Since $\phi \in L^2(\Gamma_\rho^-)$;, we have

$$\phi^-(z) = \int_0^\infty e^{-zt} g(t)\,dt, \quad \int_0^\infty e^{2\tilde{\rho}t}\left|g(t)\right|^2 dt < \infty, \tag{2.25}$$

and

$$\int_{-\infty}^{\infty} \left|\phi^-(i\eta)\right|^2 d\eta = 2\pi\int_0^\infty \left|g(t)\right|^2 dt < \infty\,. \tag{2.26}$$

Similarly

$$\phi^+(z) = \int_{-\infty}^0 e^{-zt} h(t)\,dt, \quad \int_{-\infty}^0 e^{2\tilde{\rho}|t|}\left|h(t)\right|^2 dt < \infty, \tag{2.27}$$

and

$$\int_{-\infty}^{\infty} \left|\phi^+(i\eta)\right|^2 d\eta = 2\pi \int_{-\infty}^0 \left|h(t)\right|^2 dt < \infty. \tag{2.28}$$

Condition (2.4) for ψ to be a member of Ψ_p implies that

$$\phi(i\eta) = \phi^+(i\eta) + \phi^-(i\eta), \qquad -\infty < \eta < \infty. \tag{2.29}$$

From (2.26) and (2.28) we conclude that there is a function $f \in L^2(-\infty,\infty)$ such that

$$\phi(i\eta) = \underset{A\to\infty}{\mathrm{l.i.m.}} \int_{-A}^{A} e^{-i\eta t}\phi(i\eta)d\eta. \tag{2.30}$$

and

$$f(t) = \frac{1}{2\pi} \underset{A\to\infty}{\mathrm{l.i.m.}} \int_{-A}^{A} e^{-i\eta t}\phi(i\eta)d\eta .$$

From the identity (2.29) and (2.23), (2.24) we conclude easily (since $\phi \in L^2\Gamma_\rho$) that $|\phi(z)|$ is uniformly bounded in the closed strip $|\mathrm{Re}(z)| = |\xi| < \beta$. Let C_A be the closed contour, positively oriented, consisting of the imaginary axis from $-iA$ to iA and the right half of the circle $|z| = A$. Define

$$w_A^+(t) = \frac{1}{2\pi i} \int_{D_A} e^{-zt} \frac{\phi(z)}{z+1} dz.$$

Letting $A\to\infty$, using the bounds (2.21) and (2.22) for ϕ and the Jordan lemma, we conclude

$$w(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-i\eta t} \frac{\phi(i\eta)}{i\eta+1} d\eta$$

is identically equal to zero for $t > \pi$. But the relationship of the Laplace transform to convolution shows that

$$w(t) = \int_0^t e^{-(t-s)} f(s)ds$$

and hence that

$$0 \equiv w'(t) = w(t) = f(t) \quad \text{a.e.}, \quad t > \pi$$

Thus we conclude that $f(t) = 0$ a.e., $t > \pi$. A similar argument shows $f(t) = 0$ a.e., $t < -\pi$ and we have, ferom (2.30) and the identity theorem,

$$\phi(z) = \int_{-\pi}^{\pi} e^{zt} f(t)dt, \qquad f \in L^2[-\pi,\pi],$$

valid for all complex z. Hence $\phi \in \Phi$ and the theorem is proved. For $\psi \in \Psi$ we define, for $\rho > a$ (cf. (2.3)),

$$\|\psi\|^2_\rho = \int_{\Gamma_\rho} |\psi(z)|^2 |dz| . \tag{2.31}$$

Then it is clear that the map P defined by (2.20) maps Ψ_p onto Φ. Its inverse on Φ, P^{-1}, is defined by (2.15). It is clear that both P and P^{-1} are bounded with respect to $\|\phi\|_\rho$ in Φ and $\|\psi\|_\rho$ in Ψ_p. Since Φ admits a Hilbert space structure, it follows that Ψ_p does as well.

One of the most important results of this paper has to do with the relationship between $\| \ \|_\rho$ on Ψ_p and another norm on the same space, which we refer to as $\| \ \|_\phi$. The definition of $\| \ \|_\phi$ depends on the following result.

Theorem 2.4. *Let* $\phi \in \Phi$, $\psi \in \Psi$, *and let* $f \in L[-\pi,\pi]$, $g \in L_\beta \ (-\infty,\infty)$, $\beta > a$, *be such that* $\phi = \mathfrak{F}f$, $\psi = \mathfrak{L}g$ (cf. (2.1), (2.7), (2.8)). *Then with*

$$\langle\phi,\psi\rangle \equiv \frac{1}{2\pi i}\int_{\Gamma_\rho} \phi(z)\psi(z)dz,$$

$\langle\phi,\psi\rangle$ *is independent of* ρ *for* $\rho > \beta > a$ *and we have*

$$\langle\phi,\psi\rangle = \int_{-\pi}^{\pi} f(t)g(t)dt. \tag{2.33}$$

Proof The formal argument is very simple:

$$\langle\phi,\psi\rangle = \frac{1}{2\pi i}\int_{\Gamma_\rho} \phi(z)\psi(z)dz = \frac{1}{2\pi i}\int_{\Gamma_\rho}\int_{\pi}^{\pi} e^{zt}f(t)dt\psi(z)dz$$
$$= \int_{-\pi}^{\pi} f(t)\,\frac{1}{2\pi i}\int_{\Gamma_\rho} e^{zt}\psi(z)dz\,dt = \int_{-\pi}^{\pi} f(t)g(t)dt, \tag{2.34}$$

the last identity following from (2.12). To make this argument rigorous one may define (cf. (2.11))

$$g_A(t) = \frac{1}{2\pi i}\int_{\Gamma_{\rho,A}} e^{zt}\,\psi(z)dz$$

and one immediately has

$$\int_{\pi}^{\pi} f(t)g_A(t)dt = \frac{1}{2\pi i}\int_{\Gamma_{\rho,A}} \phi(z)\psi(z)dz.$$

Since it is well known that g_A converges to g in the L^2 norm on $[-\pi,\pi]$ and $\phi(z)\psi(z) \varepsilon L^1(\Gamma_\rho)$, the desired identity follows immediately. Since the right hand side of (2.33) is independent of ρ, the same is true of the right hand side of (2.32), which is well defined for all $\rho > a$, a as defined for ψ preceding (2.4). We define a semi-norm on Ψ by

$$\|\Psi\|_\Phi = \sup_{\substack{\phi\varepsilon\Phi \\ \phi\neq 0}} \frac{|\langle\phi,\psi\rangle|}{\|\phi\|_0},$$

where

$$\|\phi\|^2_0 = \int_\infty^\infty |\phi(i\eta)|^2 |d\eta|$$

is equivalent to any of the norms $|\ |_\rho$ on Φ, $\rho > 0$. Since the Plancherel Theorem gives

$$\|\phi\|_0 = \sqrt{2\pi}\ \| {}^{-1}\phi\|_{L^2[-\pi,\pi]} = \sqrt{2\pi}\ \|f\|_{L^2[-\pi,\pi]}$$

and (2.33) obtains, we see that

$$\|\psi\|_\Phi = \frac{1}{\sqrt{2\pi}} \sup_{\substack{f\varepsilon L^2[-\pi,\pi] \\ \|f\|\neq 0}} \frac{\int_\pi^\pi f(t)g(t)dt}{\|f\|_{L^2[-\pi,\pi]}}$$

$$= \frac{1}{\sqrt{2\pi}} \|g\|_{L^2[-\pi,\pi]} \tag{2.35}$$

Thus $\|\psi\|_\Phi = 0$ if $g(t) = 0$ a.e. in $[-\pi,\pi]$. Also, since the Plancherel Theorem gives

$$\|\psi\|^2_\pi = 2\pi\left(\int_0^\infty e^{-2\rho t}|g(t)|^2 dt + \int_{-\infty}^0 e^{2\rho t}|g(t)|^2 dt\right)$$

it is clear that

$$\|\psi\|^2_\Phi = \frac{1}{2\pi}\int_{-\pi}^\pi |g(t)|^2\, dt < e^{2\pi\rho}\,\|\Psi\|^2_\rho\ . \tag{2.36}$$

A principal result of the next section will be to show that, restricted to Ψ_p, $\|\ \|_\Phi$ is a norm and an inequality in the reverse direction of (2.36) may be obtained. We make a start in this direction with

Theorem 2.5. If $\psi \varepsilon \Psi_p$ then

$$|\psi|_{\Phi} = 0 \Longrightarrow \psi(z) \equiv 0.$$

Then $\| \ \|_{\Phi}$ is a norm on Ψ_p.

Proof. If $|\psi|_{\Phi} = 0$ then for $\rho > a$.

$$\int_{\Gamma_\rho} \theta(z)\theta(z)dz = 0, \quad \nu \varepsilon \Phi. \tag{2.37}$$

Since Theorem 1.3 shows that for some $\phi \varepsilon \Phi$

$$\psi(z) = \phi(z)/p(z), \tag{2.38}$$

(2.37) becomes

$$\int_{\Gamma_\rho} \frac{\theta(z)}{p(z)} \phi(z)dz = 0, \quad \theta \varepsilon \Phi. \tag{2.39}$$

Theorem 2.3 also shows that as $\theta(z)$ runs through Φ, $\theta(z)/p(z)$ covers all of Ψ_p.

Let C be a closed contour in the complex plane not meeting any zero of $p(z)$. Then with $q(z)$ an arbitrary polynomial in z,

$$\psi_C(z) \equiv \frac{1}{2\pi i} \int_C \frac{1}{z-\zeta} \frac{q(\zeta)}{p(\zeta)} d\zeta,$$

defined for z exterior to C, is a rational function of z which belongs to Ψ_p and consequently has the form $\theta(z)/p(z)$ as in (2.39). For another contour C, just outside C and enclosing exactly the same zeros of $p(z)$, (2.39) is readily seen to imply

$$0 = \int_{\hat{C}} \phi(z)\psi_C(z)dz = \int_{\hat{C}} \frac{1}{2\pi i} \int_C \frac{1}{z-\zeta} \frac{q(\zeta)}{p(\zeta)} d\zeta \, dz$$

$$= \int_C \frac{q(\zeta)}{p(\zeta)} \frac{1}{2\pi i} \int_{\hat{C}} \frac{\phi(z)}{z-\zeta} dz \, d\zeta = \int_C q(\zeta) \frac{\phi(\zeta)}{p(\zeta)} d\zeta .$$

Since this is true for every such C and every polynomial q we conclude that $\phi(z)/p(z)$ is entire. But since $\phi(z)/p(z)$ lies in $\Psi_p \subset \Psi$, (2.4) shows that for $\rho > a$

$$\frac{\phi(z)}{p(z)} = \frac{1}{2\pi i} \int_{\Gamma_\rho} \frac{1}{z-\zeta} \frac{\phi(\zeta)}{p(\zeta)} \, d\zeta.$$

The fact that $\phi(z)/p(z)$ then shows that $\phi(z)/p(z) \equiv 0$ for z outside the closed strip bounded by Γ_ρ and thus, by the identity theorem,

$$\psi(z) = \phi(z)/p(z) \equiv 0,$$

proving the theorem.

3. The Internal and External Spaces.

As previously, let Γ_ρ denote the contour consisting of the two lines $\mathrm{Re}(\lambda) = \rho$, $\mathrm{Re}(\lambda) = -\rho$, positively oriented, and let the two halves of Γ_ρ be denoted Γ_ρ, Γ_ρ. The following theorem is well known ([H],[I]).

Theorem 3.1. Let $h_+ = h_+(\rho+i\sigma) \in L^2(\Gamma_\rho)$. Then there are uniquely defined functions $h_+(z)$, $h_+(z)$ defined and analytic in $\mathrm{Re}(z) > \rho$, $\mathrm{Re}(z) < \rho$, and lying in the Hardy spaces $H^2\{\mathrm{Re}(z) > \rho\}$, $H^2\{\mathrm{Re}(z) < \rho\}$, respectively, with boundary values in $L^2(\Gamma_\rho)$, such that

$$h_+(\rho+i\sigma) = h_+^+(\rho+i\sigma) + h_+^-(\rho+i\sigma).$$

Moreover

$$\|h_+\|^2_{L^2(\Gamma_\rho^+)} = \|h_+^+\|^2_{\Gamma_\rho^+} + \|h_+^-\|^2_{\Gamma_\rho^+} \tag{3.1}$$

While we do not offer a formal proof, it may not hurt to remind the reader that

$$h_+^+(z) = \frac{1}{2\pi i} \int_{\Gamma_\rho^+} \frac{h_+(\zeta)d\zeta}{z-\zeta}, \quad \mathrm{Re}(z) > \rho, \tag{3.2}$$

$$h_+^-(z) = \frac{1}{2\pi i} \int_{\Gamma_\rho^+} \frac{h\ (\zeta)d\zeta}{\zeta-z}, \quad \mathrm{Re}(z) < \rho, \tag{3.3}$$

the orientation of Γ_ρ^+ being upward in both cases. Moreover, there is a unique function g_+ satisfying

$$\int_{-\infty}^{\infty} e^{-2\rho t} \left| g_+(t) \right|^2 dt < \infty.$$

such that $h_+(z) = (\mathfrak{F}g)(-z)$, $\mathrm{Re}(z) = \rho$, i.e.,

$$h_+(\rho i+\sigma) = \underset{A\to\infty}{\mathrm{l.i.m.}} \int_{-A}^{A} e^{-i\sigma t} e^{-\rho t}\, g_+(t)dt\ .$$

$$= \underset{A\to\infty}{\mathrm{l.i.m.}} \int_{-A}^{A} e^{-(\rho+i\sigma)t} g_+(t)dt.$$

while $\overset{+}{h}_+(z)$, $-\overset{-}{h}_+(z)$ are the right and left Laplace transforms of g_+:

$$h_+^+(z) = \underset{A\to\infty}{\text{l.i.m.}} \int_0^A e^{-zt} g_+(t)dt, \quad Re(z) > \rho$$

$$h_+^-(z) = \underset{A\to\infty}{\text{l.i.m.}} \int_{-A}^0 e^{-zt} g_+(t)dt, \quad Re(z) < \rho .$$

In the same way, if $h_- = h_-(-\rho+i\sigma) \in L^2(\Gamma_\rho^-)$ we may decompose h_- as

$$h_-(-\rho+i\sigma) = h_-^+(-\rho+i\sigma) + h_-^-(-\rho+i\sigma)$$

where h_-^+, h_-^- lie in the Hardy spaces $H^2\{Re(z) > \rho\}$, $H^2\{Re(z) > -\rho\}$, respectively, and

$$h_-^+(z) = \frac{1}{2\pi i} \int_{\Gamma_\rho^-} \frac{h_-(\zeta)}{\zeta-z} d\zeta, \quad Re(z) > -\rho, \tag{3.4}$$

$$h_-^-(z) = \frac{1}{2\pi i} \int_{\Gamma_\rho^-} \frac{h_-(\zeta)}{z-\zeta} d\zeta, \quad Re(z) < -\rho, \tag{3.5}$$

$$\|h_-\|^2_{L^2(\Gamma_\rho^-)} = \|h_-^+\|^2_{\Gamma_\rho^-} + \|h_-^-\|^2_{\Gamma_\rho^-} . \tag{3.6}$$

Now let $h \in L^2(\Gamma_\rho)$ and let h_+, h_- be its restrictions to Γ_ρ^+, Γ_ρ^-, respectively. Define

$$\tilde{h}(z) = h_-^+(z) + h_+^-(z), \quad |Re(z)| < \rho, \tag{3.7}$$

and we have, from (3.3) and (3.4),

$$\tilde{h}(z) = \frac{1}{2\pi i} \int_{\Gamma_\rho} \frac{h(\zeta)}{\zeta-z} d\zeta. \tag{3.8}$$

We will refer to $\tilde{h}$ as the "internal part of h relative to Γ_ρ". (If h is defined originally on a set which includes Γ_ρ for various values of ρ it is necessary to refer to the particular Γ_ρ in question. If Γ_ρ is understood, we will simply refer to h as the "internal" part of h.) We will write (3.8) as

$$\tilde{h} = \tilde{T}h$$

and designate

$$\tilde{H}^2(\Gamma_\rho) = \{\tilde{T}h \mid h \in L^2(\Gamma_\rho)\} = \tilde{T}(L^2(\Gamma_\rho))$$

as the "internal Hardy space" (relative to Γ_ρ). We define

$$\hat{h}(z) = h_+^+(z) - h_-^+(z), \quad \mathrm{Re}(z) > \rho,$$

$$\hat{h}(z) = h_-^-(z) - h_+^-(z), \quad \mathrm{Re}(z) < \rho,$$

and we have, for $|\mathrm{Re}(z)| > \rho$

$$\hat{h}(z) = \frac{1}{2\pi i} \int_{\Gamma_\rho} \frac{h(\zeta)}{z-\zeta} d\zeta, \tag{3.9}$$

as may be readily verified. We write (3.9) as

$$\hat{h} = \hat{T}h$$

and refer to $\hat{h}$ as the *external part of* h (relative to Γ_ρ). The space

$$\hat{H}^2(\Gamma_\rho) = \{\hat{T}h \mid h \in L^2(\Gamma_\rho)\} = \hat{T}(L^2(\Gamma_\rho))$$

is designated as the "external Hardy space" (relative to Γ_ρ). It is clear that

$$\tilde{h}(z) + \hat{h}(z) = h(z), \quad z \in \Gamma_\rho \tag{3.10}$$

so that

$$\tilde{T} + \hat{T} = I .$$

It is easy to see that $\tilde{T}$ and $\hat{T}$ are both projections, onto $\tilde{H}^2(\Gamma_\rho)$, $\hat{H}^2(\Gamma_\rho)$, respectively, but that they are not mutually orthogonal. Using the properties of the Hardy spaces one may see that

$$\|h_+^-\|_{L^2(\Gamma_\rho^-)} < \|h_+^-\|_{L^2(\Gamma_\rho^+)}$$

$$\|h_-^+\|_{L^2(\Gamma_\rho^+)} < \|h_-^+\|_{L^2(\Gamma_\rho^-)}$$

and from this we have, using (3.1), (3.6)

$$\|\tilde{h}\|_{L^2(\Gamma_\rho)} \le \|h_-^+\|_{L^2(\Gamma_\rho)} + \|h_+^-\|_{L^2(\Gamma_\rho)}$$

$$\le 2\left(\|h_+^-\|_{L^2(\Gamma_\rho^-)} \le \|h_+^-\|_{L^2(\Gamma_\rho^+)}\right) \le 2\,\|h\|_{L^2(\Gamma_\rho)} \qquad (3.11)$$

$$\|\hat{h}\|_{L^2(\Gamma_\rho)} \le \|h_+^+\|_{L^2(\Gamma_\rho^+)} + \|h_+^-\|_{L^2(\Gamma_\rho^+)}$$

$$+ \|h_-^-\|_{L^2(\Gamma_\rho^-)} \le \|h_+^-\|_{L^2(\Gamma_\rho^-)}$$

$$\le \|h_+^+\|_{L^2(\Gamma_\rho^+)} \le \|h_-^+\|_{L^2(\Gamma_\rho^-)} + \|h_-^-\|_{L^2(\Gamma_\rho^-)} + \|h_+^-\|_{L^2(\Gamma_\rho^+)}$$

$$\le \sqrt{2}\left(\|h_+\|_{L^2(\Gamma_\rho^+)} + \|h_+\|_{L^2(\Gamma_\rho^-)}\right) \le 2\,\|h\|_{L^2(\Gamma_\rho)} \,. \qquad (3.12)$$

On the other hand (3.10) gives

$$\|h\|_{L^2(\Gamma_\rho)} \le \|\tilde{h}\|_{L^2(\Gamma_\rho)} + \|\hat{h}\|_{L^2(\Gamma_\rho)} \,. \qquad (3.13)$$

A final point in our elucidation of the properties of $H^2(\Gamma_\rho)$ and $H^2(\Gamma_\rho)$ is this: if $h \in L^2(\Gamma_\rho)$ and $\tilde{h}$, $\hat{h}$ are its internal and external parts, then $\|\tilde{h}\|_{L^2(\Gamma_\sigma)}$ and $\|h\|_{L^2(\Gamma_\tau)}$ can each be uniformly bounded in terms of $\|h\|_{L^2(\Gamma_\rho)}$, provided $0 < \sigma < \rho < \tau$. Such a result is easily obtained using arguments of much the same type as those used above.

Our next task is to identify the Hilbert spaces Φ and Ψ with subspaces of $\tilde{H}^2(\Gamma_\rho)$, $\hat{H}^2(\Gamma_\rho)$, respectively.

Proposition 3.2: Let $\phi \in \Phi$, so that $\phi|\Gamma_\rho \in L^2(\Gamma_\rho)$. Then $\tilde{\phi} = \phi$ so that $\phi|\Gamma_\rho = \tilde{T}\Phi_{\Gamma_\rho} \subseteq \tilde{T}(L^2(\Gamma_\rho)) = \tilde{H}^2(\Gamma_\rho)$.

Proof. This follows from (3.8) and the fact that (2.14) is valid for all $\phi \in \Phi$.

Proposition 3.3. Let $\psi \varepsilon \Psi$, so that $\psi | \Gamma_\rho \varepsilon L^2(\Gamma_\rho)$ for $\rho > a$ (cf. (2.5)). Then $\hat{\psi} \varepsilon \psi$ so that $\Psi | \Gamma_\rho = \hat{T}\Psi | \Gamma_\rho \subseteq \hat{T}(L^2(\Gamma_\rho)) = H^2(\Gamma_\rho)$.

Proof. This follows from (3.9) and the fact that (2.4) is valid for all $\psi \varepsilon \Psi$.

We see then that for each cardinal function p, the map P defined by (2.20), and its inverse, P^{-1}, are external $\to$ internal and internal $\to$ external maps, respectively, defined on $L^2(\Gamma_\rho)$.

The following theorem is the basic result concerning "interpolation" of a function $f \varepsilon \tilde{H}^2(\Gamma_\rho)$ by a function $\phi \varepsilon \Phi$ on the zero set, Z_p, of a cardinal function p.

Theorem 3.4. Let p be a cardinal function and let $\rho > \alpha$ (cf. (1.4)). Let $h \varepsilon \tilde{H}^2(\Gamma_\rho)$; thus h may be extended into $\mathrm{Int}(\Gamma_\rho)$ via

$$h(z) = \frac{1}{2\pi i} \int_{\Gamma_\rho} \frac{h(\zeta)}{\zeta - z}\, d\zeta. \tag{3.14}$$

Then there is a unique $\phi \varepsilon \Phi$ such that $(\phi - f)/p$ is holomorphic in $\mathrm{Int}(\Gamma_\rho)$. Moreover, there is a positive K, independent of h, such that

$$|\phi|_\rho \le K |h|_{L^2(\Gamma_\rho)}. \tag{3.15}$$

Remarks. The term "extended" has a technical sense here because $h | \Gamma_\rho$ is the limit in the L^2-norm, of $h | \Gamma_{\rho'}$, $\rho' < \rho$, as $\rho' \to \rho$.

The term "interpolation on Z_p" is used advisedly since for each zero λ of p, of multiplicity μ, $\phi(\lambda), \phi'(\lambda), \cdots, \phi^{\mu-1}(\lambda)$ must agree with $h(\lambda), h'(\lambda), \cdots, h^{\mu-1}(\lambda)$, respectively.

Proof of Theorem 3.4. The uniqueness is quite straightforward. If ϕ_1, ϕ_2 were two such functions in Φ, we would have

$$\frac{\phi_1-\phi_2}{p} = \frac{\phi_1-f}{p} = \frac{\phi_2-f}{p}$$

on the one hand in Ψ_p by Theorem 2.2, and, on the other, holomorphic in $\mathrm{Int}(\Gamma_\rho)$. The formula (2.4), valid for $\psi \in \Psi$ and z external to Γ_ρ gives

$$\frac{\phi_1(z)-\phi_2(z)}{p(z)} = \frac{1}{2\pi i}\int_{\Gamma_\rho} \frac{\phi_1(\zeta)-\phi_2(\zeta)}{(z-\zeta)p(\zeta)}\, d\zeta \ .$$

The properties of $\phi \in \Phi$ and p, together with the holomorphicity of $(\phi_1-\phi_2)/p$ in $\mathrm{Int}(\Gamma_\rho)$, show that the integral converges and converges to zero. Thus

$$\phi_1(z) = \phi_2(z), \quad \left|\mathrm{Re}(z)\right| > \rho$$

and extends, using the identity theorem, to all z.

For the existence, we let Γ_σ be a contour similar to Γ_ρ but with $\alpha < \sigma < \rho$. For $\left|\mathrm{Re}(z)\right| > \sigma$ we define

$$\psi(z) = \frac{1}{2\pi i}\int_{\Gamma_\sigma} \frac{h(\zeta)d\zeta}{(z-\zeta)p(\zeta)} = \hat{T}\left(\frac{h}{p}\right) . \tag{3.16}$$

The integral is convergent; p is bounded below on Γ_ρ and the square integrability of h on Γ_σ is a consequence of its membership in $\tilde{H}^2(\Gamma_\sigma)$. Then, still for $\left|\mathrm{Re}(z)\right| > \sigma$, we define

$$\phi(z) = p(z)\psi(z) = p(z)\,\frac{1}{2\pi i}\int_{\Gamma_\sigma} \frac{h(\zeta)d\zeta}{(z-\zeta)p(\zeta)} \ . \tag{3.17}$$

Then we define $\tilde{\phi}(w)$, $\left|\mathrm{Re}(w)\right| < \rho$, in agreement with (2.14), by

$$\tilde{\phi}(w) = \frac{1}{2\pi i}\int_{\Gamma_\rho} \frac{\phi(z)}{z-w}\, dz = \tilde{T}(\phi). \tag{3.18}$$

From (3.16), $\psi \in \hat{H}^2(\Gamma_\sigma)$, so $\psi \in L^2(\Gamma_\rho)$. From the properties of p, ϕ, defined by (3.17), is in $L^2(\Gamma_\rho)$ and then $\tilde{\phi} \in \tilde{H}^2(\Gamma_\rho)$.

Let w satisfy $\sigma < \left|\mathrm{Re}(w)\right| < \rho$. Then

$$\tilde{\phi}(w) = \frac{1}{2\pi i}\int_{\Gamma_\rho} \frac{\phi(z)}{z-w}\,dz = \frac{-1}{4\pi^2}\int_{\Gamma_\rho} \frac{p(z)}{z-w}\int_{\Gamma_\sigma} \frac{h(\zeta)d\zeta}{(z-\zeta)p(\zeta)}\,dz$$

$$= \frac{-1}{4\pi^2}\int_{\Gamma_\rho} \frac{h(\zeta)}{(w-\zeta)p(\zeta)}\int_{\Gamma_\rho}\left|\frac{p(z)}{z-w} - \frac{p(z)}{z-\zeta}\right| dz\,d\zeta$$

$$= p(w)\frac{1}{2\pi i}\int_{\Gamma_\sigma} \frac{h(\zeta)d\zeta}{(w-\zeta)p(\zeta)} - \frac{1}{2\pi i}\int_{\Gamma_\sigma} \frac{h(\zeta)}{w-\zeta}\,d\zeta \qquad (3.19)$$

Since $h \in \tilde{H}^2(\Gamma_\rho)$, we have also $h \in \tilde{H}^2(\Gamma_\sigma)$. Since w is exterior to Γ_σ

$$\frac{1}{2\pi i}\int_{\Gamma_\sigma} \frac{h(\zeta)}{w-\zeta}\,d\zeta = 0$$

and we therefore have, from (3.19),

$$\tilde{\phi}(w) = p(w)\frac{1}{2\pi i}\int_{\Gamma_\sigma} \frac{h(\zeta)}{(w-\zeta)p(\zeta)} = \phi(w).$$

It follows that $\tilde{\phi}(w)$ provides an analytic continuation of ϕ, as defined by (3.17), into the region $|\mathrm{Re}(z)| < \rho$. Thus ϕ is entire. That $\phi \in \Phi$ may be deduced from $\phi(z) = \tilde{\phi}(z)$, $\phi \in \tilde{H}^2(\Gamma_\rho)$, $|\mathrm{Re}(z)| < \rho$, together with $\phi(z) = p(z)\psi(z)$, $|\mathrm{Re}(z)| > \sigma$. In particular, (2.14) follows from (3.18) as soon as $\phi = \tilde{\phi}$ has been established.

There may be some question about the change of the order of integration in (3.19). Let $\Gamma_{\rho,A}$ be defined as in (2.11) and let $R_{\rho,A}$ be defined as preceding (2.18). Since $\psi(z)$ as defined by (3.16) is in $L^2(\Gamma_\rho)$

$$-\frac{1}{4\pi^2}\int_{\Gamma_\rho} \frac{p(z)}{z-w}\int_{\Gamma_\sigma} \frac{h(\zeta)d\zeta}{(z-\zeta)p(\zeta)}\,dz = -\frac{1}{4\pi^2}\lim_{A\to\infty}\int_{\Gamma_{\rho,A}} \frac{p(z)}{z-w}\int_{\Gamma_\sigma} \frac{h(\zeta)d\zeta}{(z-\zeta)p(\zeta)}\,dz$$

$$= \frac{1}{4\pi^2}\lim_{A\to\infty}\int_{\Gamma_\sigma} \frac{h(\zeta)}{(w-\zeta)p(\zeta)}\int_{\Gamma_{\rho,A}}\left|\frac{p(z)}{z-w} - \frac{p(z)}{z-\zeta}\right| dz\,d\zeta \qquad (3.20)$$

because

$$\frac{p(z)h(\zeta)}{(z-w)(z-\zeta)p(\zeta)}$$

is integrable for $\zeta \in \Gamma_\sigma$, $z \in \Gamma_{\rho,A}$. Then we note, since Γ_σ is interior to Γ_ρ, that with $p_A(\zeta) = p(\zeta)$, $|\zeta| < A$; $p_A(\zeta) = 0$, $|\zeta| > A$, and $A > |w|$,

$$\frac{1}{2\pi i}\int_{R_{\rho,A}}\left|\frac{p(z)}{z-w}-\frac{p(z)}{z-\zeta}\right|dz = p(w) - p_A(\zeta).$$

Since $h(\zeta)/(z-\zeta)$ is integrable on Γ_σ

$$\lim_{A\to\infty}\int_{\Gamma_\sigma}\frac{h(\zeta)p_A(\zeta)d\zeta}{(w-\zeta)p(\zeta)} = \int_{\Gamma_\sigma}\frac{h(\zeta)}{w-\zeta}d\zeta\ .$$

Since it is easily established that

$$\lim_{A\to\infty}\left|\left(\int_{R_{\rho,A}}-\int_{\Gamma_{\rho,A}}\right)\left[\frac{p(z)}{z-w}-\frac{p(z)}{z-\zeta}\right]dz\right| = 0,$$

we conclude that the last expression in (3.20) converges, as A tends to ∞, to the corresponding expression in (3.19), which is all we need.

Again for $\sigma < |\mathrm{Re}(w)| < \rho$ we note that

$$\frac{\phi(w)-h(w)}{p(w)} = \frac{1}{2\pi i}\int_{\Gamma_\sigma}\frac{h(\zeta)d\zeta}{(w-\zeta)p(\zeta)} - \frac{h(w)}{p(w)}\ ,$$

But one shows quite readily that

$$\frac{h(w)}{p(w)} = \frac{1}{2\pi i}\int_{\Gamma_\rho}\frac{h(\zeta)d\zeta}{(w-\zeta)p(\zeta)} + \frac{1}{2\pi i}\int_{\Gamma_\sigma}\frac{h(\zeta)d\zeta}{(w-\zeta)p(\zeta)}$$

since $h(\zeta)/p(\zeta)$ is holomorphic in the region $\sigma < |\mathrm{Re}(\zeta)| < \rho$. It follows that

$$\frac{\phi(w)-h(w)}{p(w)} = \frac{1}{2\pi i}\int_{\Gamma_\rho}\frac{h(\zeta)d\zeta}{(\zeta-w)p(\zeta)}\ .$$

Since the right hand side defines a function which is holomorphic for $|\mathrm{Re}(w)| < \rho$, the left hand side must be holomorphic there as well.

Finally, there is the bound (3.15). This follows immediately from (3.17) and (3.18). For $h(\zeta)/p(\zeta)$ lies in $L^2(\Gamma_\rho)$, and, since

$$\frac{1}{2\pi i}\int_{\Gamma_\sigma}\frac{h(\zeta)d\zeta}{(w-\zeta)p(\zeta)} = \psi(z)$$

is the external part of this function, we see that $\psi|\Gamma_\rho \in L^2(\Gamma_\rho)$ and may be bounded in terms of $|h|_{L^2(\Gamma_\sigma)}$, using the fact that p is bounded below on

Γ_σ for $\sigma > \alpha$. Then we note that $\tilde{\phi}$, defined by (3.18), is the internal part of $p\psi$ relative to Φ_ρ and, using the fact that p is bounded above on Γ_ρ, we bound $\|\tilde{\phi}\|_\rho = |\phi|_\rho$ in terms of $|\psi|_{L^2(\Gamma_\rho)}$, which in turn is bounded in terms of $|h|_{L^2(\Gamma_\sigma)}$, and that may be bounded in terms of $|h|_{L^2(\Gamma_\rho)}$, which completes the proof.

<u>Corollary 3.5.</u> <u>Let</u> $\tilde{h} \in \tilde{H}^2(\Gamma_\rho)$ <u>and let</u> ϕ <u>be constructed as in</u> (3.17). <u>Then</u> $\phi/p \in \Psi_p$ <u>and for every</u> $\theta \in \Phi$.

$$\frac{1}{2\pi i}\int_{\Gamma_\rho} \frac{\theta(z)\tilde{h}(z)dz}{p(z)} = \frac{1}{2\pi i}\int_{\Gamma_\rho} \frac{\theta(z)\phi(z)dz}{p(z)} . \tag{3.22}$$

<u>Proof.</u> The conclusion $\phi(z)/p(z) \in \Psi_p$ follows from Theorem 2.2 since $\phi \in \Phi$. For $\alpha < \sigma < \rho$ we have

$$\frac{1}{2\pi i}\int_{\Gamma_\rho} \frac{\theta(z)\phi(z)dz}{p(z)} = \frac{1}{4\pi 2}\int_{\Gamma_\rho} \theta(z) \int_{\Gamma_\sigma} \frac{\tilde{h}(\zeta)}{(z-\zeta)p(\zeta)}\, d\zeta\, dz$$

$$= -\frac{1}{4\pi 2}\int_{\Gamma_\sigma} \frac{\tilde{h}(\zeta)}{p(\zeta)} \int_{\Gamma_\rho} \frac{\theta(z)}{z-\zeta}\, dz\, d\zeta = \frac{1}{2\pi i}\int_{\Gamma_\sigma} \frac{\tilde{h}(\zeta)\theta(\zeta)d\zeta}{p(\zeta)}$$

$$= \frac{1}{2\pi i}\int_{\Gamma_\rho} \frac{\theta(z)\tilde{h}(z)dz}{p(z)} .$$

The change of order of integration is established in much the same way as in the preceding theorem. The last identity follows from the analyticity of $\theta h/p$ in the region $\sigma < |\mathrm{Re}(z)| < \rho$ together with by now familiar estimates on the integrand as $|\mathrm{Im}(z)| \to \infty$.

<u>Theorem 3.6.</u> <u>For any cardinal function</u> p <u>there is a positive number</u> K_p <u>such that for all</u> $\psi \in \Psi_p$ (<u>and</u> $\rho > \alpha$, cf. (1.4))

$$\|\psi\|_\rho \leq K_p \|\psi\|_\phi . \tag{3.23}$$

<u>Hence</u> $\|\ \|_\rho$ <u>and</u> $\|\ \|_\phi$ <u>are equivalent on</u> Ψ_p.

<u>Proof.</u> Let $\psi \in \Psi_\rho$ and let $\rho > \alpha$. Then

$$\|\psi\|^2_\rho = \int_{\Gamma_\rho} |\psi(z)|^2 \, |dz| = \frac{1}{2\pi i} \int_{\Gamma_\rho} \psi(z)h(z)dz$$

where

$$h(z) = 2\pi i\overline{\psi(z)} \frac{|dz|}{dz} = \begin{cases} \overline{2\pi\psi(z)}, & z \in {}^{+}\Gamma_\rho \\ -2\pi\overline{\psi(z)}, & z \in \Gamma_\rho^- . \end{cases}$$

Clearly $h \in L^2(\Gamma_\rho)$ and hence can be written

$$h(z) = \tilde{h}(z) + \hat{h}(z), \quad \tilde{h} \in \tilde{H}^2(\Gamma_\rho), \quad \hat{h} \in \hat{H}^2(\Gamma\rho).$$

We claim that

$$\int_{\Gamma_\rho} \psi(z)\hat{h}(z)dz = 0.$$

For if $\tau > \rho$ we can easily show that

$$\int_{\Gamma_\rho} \psi(z)\hat{h}(z)dz = \int_{\Gamma_\tau} \psi(\zeta)\hat{h}(\zeta)d\zeta = (\text{cf.}(3.9))$$

$$= \int_{\Gamma_\tau} \psi(\zeta) \frac{1}{2\pi i} \int_{\Gamma_\rho} \frac{\hat{h}(z)}{z-\zeta} \, dz \, d\zeta$$

$$= \int_{\Gamma_\rho} \hat{h}(z) \frac{1}{2\pi i} \int_{\Gamma_\rho} \frac{\psi(\zeta)}{z-\zeta} d\zeta \, dz = 0$$

because z on Γ_ρ is external to the region $\mathrm{Re}(\zeta) > \sigma$ in which $\psi(\zeta)$ is analytic and $|\psi(\zeta)| \to 0$ uniformly as $|\zeta| \to \infty$. The last property is any easy consequence of

$$\psi(\zeta) = \frac{1}{2\pi i} \int_{\Gamma_\rho} \frac{\psi(w)dw}{\zeta-w}$$

valid for $\mathrm{Re}(\zeta) > \rho$. Thus, using Corollary 3.5 with $\psi(\zeta)$ expressed as

$$\psi(\zeta) = \theta(\zeta)/p(\zeta), \quad \theta \in \Phi,$$

and ϕ related to h as in Theorem 3.4 and Corollary 3.5

$$\|\psi\|_\rho^2 = \frac{1}{2\pi i} \int_{\Gamma_\rho} \psi(z)\tilde{h}(z)dz = \frac{1}{2\pi i} \int_{\Gamma_\rho} \frac{\theta(z)h(z)dz}{p(z)} =$$

(cf.(3.22))
$$\frac{1}{2\pi i} \int_{\Gamma_\rho} \frac{\nu(z)\phi(z)dz}{p(z)} = \langle\phi,\psi\rangle$$

$$< \|\phi\|_0 \quad \|\psi\|_\Phi < B \|\phi\|_\rho \quad \|\psi\|_\Phi \quad , \tag{3.24}$$

with B independent of ψ since $\|\phi\|_0$ and $\|\phi\|_\rho$ are equivalent. But $\|h\|_{L^2(\Gamma_\rho)}$ can be bounded in terms of $\|\psi\|_\rho$ (see (3.11)) and (cf. (3.15)) $\|\phi\|_\rho$ can be bounded in terms of $\|h\|_{L^2(\Gamma_\rho)}$. Hence there is a positive number K_p, depending only on p, such that

$$\|\phi\|_\rho < K_p \|\psi\|_\rho \quad .$$

Using this in (3.24) we have (3.23) and the proof is complete.

Thus we see that Ψ_p, equipped with any of the norms $\| \ \|_\rho$, $\rho > \alpha$, is a representation of Φ', the dual space of Φ, duality relationship being expressed by $\langle\phi,\psi\rangle$, $\phi \in \Phi$, $\psi \in \Psi_p$.

A representation of the dual space Φ' independent of p may be obtained in the following way. Let $\psi \in \Psi$. Let p be a cardinal function and let $\rho > \alpha$ (cf. (1.4)). Let

$$\psi(z) = h(z)/p(z)$$

and we see, since p is bounded below on Γ_ρ, that $h \in L^2(\Gamma_\rho)$. Write $h = \tilde{h} + \hat{h}$, $\tilde{h} \in \hat{H}^2(\Gamma_\rho)$, $\hat{h} \in \hat{H}^2(\Gamma_\rho)$, and we see that for every $\theta \in \Phi$

$$\langle\theta,\psi\rangle = \frac{1}{2\pi i} \int_{\Gamma_\rho} \frac{\theta(z)}{p(z)} h(z)dz = \frac{1}{2\pi i} \int_{\Gamma_\rho} \frac{\theta(z)\tilde{h}(z)dz}{p(z)}$$

$$\pm \frac{1}{2\pi i} \int_{\Gamma_\rho} \theta(z) \frac{\phi(z)}{p(z)} dz$$

with ϕ constructed from $\tilde{h}$ as in (3.17). We know that $\phi/p \in \Psi_p \quad \Psi$. Define equivalence classes in Ψ by

$$\{\psi\} = \{\hat{\psi} \in \Psi \,|\, \langle\phi,\psi\rangle = \langle\phi,\hat{\psi}\rangle \text{ for all } \phi \in \Phi\}.$$

Then

$$\|\{\psi\}\|_{\Phi} = \|\psi\|_{\Phi} \qquad \psi \in \{\psi\}$$

defines a norm $|\ \ |_{\Phi}$, on Ψ. Given a cardinal function p, each equivalence class $\{\psi\}$ contains exactly one representative ϕ/p from Ψ_p and

$$\|\{\psi\}\|_{\Phi} = \|\phi/p\|_{\Phi} ,$$

so the map

$$\{\psi\} \to \phi/p \in \{\psi\}$$

is an isometry between Ψ and Ψ_p relative to $\|\{\psi\}\|_{\Phi}$ and $\|\phi/p\|_{\Phi}$.

Defining

$$\mathcal{E} = \{\{\psi\} \,|\, \psi \in \Psi\},$$

$\mathcal{E}$ with the norm $\|\ \ \|_{\Phi}$ is a representation of Φ'.

There are other subspaces of Ψ, besides the space Ψ_p which we have described, for which the result of Theorem 3.6 remains valid. Let $a > 0$ and let

$$Z \subset \{z \,|\, |\mathrm{Re}(z)| < a\}$$

consist of a sequence of numbers:

$$Z = \{z_k \,|\, -\infty < k < \infty\}$$

with the property that

$$\inf_k (\mathrm{Im}(z_k) - \mathrm{Im}(z_{k-1})) = d > 1.$$

Let Ψ_Z be the closed span in Ψ of the function

$$\psi_k(z) = \frac{1}{z-z_k}, \quad -\infty < k < \infty .$$

We have, of course, for $|\mathrm{Re}(z)| > a$,

$$\psi_k(z) = (\mathcal{L}e_k)(z), \quad e_k(t) = e^{z_k t} .$$

Results due to Ingham [J] and Duffin and Schaeffer [F] show that for each sequence $\{c_k\}$ ℓ^2 the series

$$g(t) = \sum_{k=-\infty}^{\infty} c_k e^{z_k t} \tag{3.25}$$

converges in $L^2[-\pi,\pi]$, and there are positive numbers C, c, depending only on a,d, such that

$$c^{-2} |g|^2_{L^2[-\pi,\pi]} < \sum_{k=-\infty}^{\infty} |c_k|^2 < C^2 |g|^2_{L^2[-\pi,\pi]}. \tag{3.26}$$

We know that $\|\psi\|_\Phi$ is equivalent to $\|g\|_{L^2[-\pi,\pi]}$. Therefore for some other numbers $\hat{C}$, $\hat{c}$, also positive, with

$$\psi(z) = \sum_{k=-\infty}^{\infty} \frac{c_k}{z-z_k}, \tag{3.27}$$

this series is convergent in Ψ with respect to $\| \ \|_\Phi$ and

$$\hat{c}^{-2} \|\psi\|^2_\Phi < \sum_{k=-\infty}^{\infty} |c_k|^2 < \hat{C}^2 \|\psi\|^2_\Phi . \tag{3.28}$$

For each integer ℓ, (3.25), (3.26) show that for $t \in [-\pi,\pi]$

$$g_\ell(t) = g(t+2\ell\pi) = \sum_{k=-\infty}^{\infty} (c_k e^{2\ell\pi z_k}) e^{z_k t} \tag{3.29}$$

converges in $L^2[-\pi,\pi]$ and

$$c^{-2} \|g_\ell\|^2_{L^2[-\pi,\pi]} < \sum_{k=-\infty}^{\infty} |c_k e^{2\ell\pi z_k}|^2 < C^2 \|g_\ell\|^2_{L^2[-\pi,\pi]},$$

so that

$$e^{-4\ell\pi a} c^{-2} \|g\|^2_{L^2[-\pi,\pi]} < \sum_{k=-\infty}^{\infty} |c_k|^2 < e^{4\ell\pi a} C^2 \|g_\ell\|^2_{L^2[-\pi,\pi]}.$$

From this we conclude that (3.29) defines a function $g(t)$ on $(-\infty,\infty)$ such that for any $\rho > a$, $g \in L_\rho(-\infty,\infty)$, i.e.,

$$e^{-\rho t} g(t) \in L^2[0,\infty),$$
$$e^{\rho t} g(t) \in L^2(-\infty,0],$$

Then, clearly, the series also converges to ψ in Ψ with respect to $\| \ \|_\rho$ and there are positive numbers $\tilde{C}$, $\tilde{c}$, depending on ρ, such that

$$\tilde{c}^{-2} \|\psi\|_\rho^2 < \sum_{k=-\infty}^{\infty} |c_k|^2 < \tilde{C}^2 \|\psi\|_\rho^2 \tag{3.30}$$

Then from comparison of (3.28) and (3.30) we have

Proposition 3.7. The subspace $\Psi_z \subset \Psi$ consists precisely of series (3.27) with $c_k \in \ell^2$, the norms $\| \ \|_\Phi$ and $\| \ \|_\rho$ are equivalent on Ψ_z is closed with respect to the topologies derived from $\| \ \|_\rho$. Moreover, the map

$$T : \{c_k\} \in \ell^2 \to \psi \in \Psi$$

defined by (3.27) is bounded and boundedly invertible (on Ψ_z) with respect to $\|\{c_k\}\|_{\ell^2}$ and either $\|\psi\|_\Phi$ or $\|\psi\|_\rho$, the bound depending only on d and a.

This result will play an important role in the next section.

4. "Regular" Nonharmonic Fourier Series in $L^2[-\pi,\pi]$.

We have defined in (2.7) the Laplace transform of a locally square integrable function g in $L^2_\rho(-\infty,\infty)$(cf.(2.6)),

$$\psi(z) = (\ g)(z).$$

Then with $\phi(z) = (\mathfrak{F}f)(z)$, $f \varepsilon L^2[-\pi,\pi]$, we have seen that

$$\langle\phi,\psi\rangle = \int_{-\pi}^{\pi} f(t)g(t)dt. \tag{4.1}$$

As a consequence $\|\psi\|_\phi$ is equivalent to $\|g\|_{L^2[-\pi,\pi]}$. When ψ is restricted to lie in Ψ_p, we know that $\|\psi\|_\phi$ is equivalent to $\|\psi\|_\rho$, which in turn is equivalent to

$$\left[\int_\infty^0 e^{2\rho t}\,|\,g(t)\,|^2\,dt + \int_0^\infty e^{-2\rho t}\,|\,g(t)\,|^2\,dt\right]^{1/2}$$
$$\equiv |\,g\,|_{L^2_\rho(-\infty,\infty)}.$$

We see then that, for $g \varepsilon \mathcal{L}^{-1}\ \Psi_p$, $\|g\|_\rho$ is equivalent to $\|g\|_{L^2[-\pi,\pi]}$.

<u>Proposition 4.1.</u> <u>If</u> p <u>is a cardinal function,</u> $\mathcal{L}^{-1}\Psi_p$ <u>is dense in</u> $L^2[-\pi,\pi]$.

<u>Proof.</u> Since (2.33) is valid for each $f \varepsilon L^2[-\pi,\pi]$ we need only show that

$$\langle\phi,\psi\rangle = 0, \quad \text{for all} \quad \psi \varepsilon \Psi_p \tag{4.2}$$

implies $\phi = 0$. But for $\psi \varepsilon \Psi_p$

$$\psi(z) = \theta(z)/p(z), \quad \theta \varepsilon \Phi,$$

and then for $\rho > \alpha$

$$\langle\phi,\psi\rangle = \frac{1}{2\pi i}\int_{\Gamma_\rho}\frac{\phi(z)\theta(z)}{p(z)}\,dz = \langle\theta,\frac{\phi}{p}\rangle. \tag{4.3}$$

If (4.2) is true, Theorems 2.2, 2.3 show (4.3) equals zero for all $\theta \varepsilon \Phi$ and then Theorem 2.5 shows that $\phi(z)/p(z) \equiv 0$ which implies $\phi(z) \equiv 0$ and we have our result.

We will see now that this proposition is really a statment about the completeness of certain complex exponentials in the space $L^2[-\pi,\pi]$.

Let the points in Z_p, the zero set of the cardinal function p, (Z_p may be shown to be non-empty quite easily using familiar theorems (cf. [K]) about entire functions) be indexed as z_k, $k \in K$, where K is a countable index set, and let μ_k be the mulciplicity of z_k as a zero of p. We denote by E_p the set of generalized exponentials

$$\{e^{z_k t}, te^{z_k t}, \cdots, t^{\mu_k - 1} e^{z_k t} \mid z_k \in Z_p\} \tag{4.4}$$

and by $[E_p]$ be span of these functions in $L^2[-\pi,\pi]$. It will be recognized immediately that

$$[E_p] = \mathcal{L}^{-1}(R_p)$$

where R_p is the subspace of Ψ_p *consisting of rational functions*

$$\rho(z) = \frac{\sigma(z)}{\tau(z)}$$

where $\sigma(z)$, $\tau(z)$ are polynomials in z with $\deg \sigma < \deg \tau$ and $p(z)\rho(z) \in \Phi$ is entire. The completeness of E_p in $L^2[-\pi,\pi]$, i.e., the fact that

$$\overline{[E_p]} = L^2[-\pi,\pi]$$

is equivalent to the denseness of R_p in Ψ_p by virtue of the remarks which we have made above. Now R_p is complete in Ψ_p just in case, for $\phi \in \Phi$

$$\langle \phi,\rho \rangle = 0, \quad \rho \in R_p, \implies \phi = 0. \tag{4.6}$$

That is the case in just the argument already given in Theorem 2.5 with the rational functions ψ_C in place of ρ: if (4.6) were true then ϕ/p would be entire and hence zero, so that $\phi = 0$. Thus we have

<u>Theorem 4.2.</u> <u>If</u> p <u>is a cardinal function then</u> R_p <u>is dense in</u> Ψ_p; <u>equivalently,</u> (4.5) <u>is true, i.e.,</u> E_p <u>is complete in</u> $L^2[-\pi,\pi]$.

This is, of course, not a new result (see, e.g. [E], [F], [K]) and is included here simply to make our presentation self-contained.

The functions (4.4) comprising E_p have the property of strong linear independence in $L^2[-\pi,\pi]$ just in case no such function lies in the closed span of the other elements of E_p; in the context of Φ and Ψ_p this is equivalent to the constructibility of the Lagrange functions $g_{k,\nu} \in \Phi$, $k \in K$, $0 < \nu < \mu_k$, with the property

$$q_{k,\nu}^{(j)}(z_\ell) = \begin{cases} 0, & \ell \neq k \\ 0, & \ell = k, \quad j \neq \nu \\ 1, & \ell = k, \quad j = \nu. \end{cases}$$

Since it is easy to see that these can be constructed in the form (the $c_{\nu,n}$ are complex scalars)

$$q_{k,\nu}(z) = p(z) \sum_{n=1}^{\mu-\nu} \frac{c_{\nu,n}}{(z-z_k)^n}$$

we will regard this strong linear independence as established.

We see, therefore, that E_p forms a basis for $L^2[-\pi,\pi]$ in the sense of constituting a complete, strongly independent set. A decidedly more ambitious enterprise is to give conditions sufficient in order that E_p should be a Schauder basis for $L^2[-\pi,\pi]$, i.e., denoting the elements (4.4) of E_p by $e_{k,\nu}$, $k \in K$, $0 < \nu < \mu$, that each $g \in L^2[-\pi,\pi]$ should have a unique convergent expansion

$$g = \sum_{k \in K} \sum_{\nu=0}^{\mu-1} g_{k,\nu}\, e_{k,\nu}\,, \tag{4.7}$$

the $g_{k,\nu}$ being complex scalars. The uniqueness is already in hand, actually, because it is easy to see that if it were violated for some $g \in L^2[-\pi,\pi]$ the

$e_{k,\nu}$ could not be strongly independent. Thus it is the existence of a convergent series as shown in (4.7) which is the main question. It appears to us that the most usable sufficient condition, stated in the context of our development, is the following. We recall ([N]) that p is <u>almost periodic</u> in a strip $|\operatorname{Re}(z)| < \beta$ just in case for each $\varepsilon > 0$ there is a positive number $\ell = \ell(\varepsilon,\beta)$ such that in each interval $[\zeta,\zeta+L]$ of the real axis of length $L > \ell$ there is at least one number η such that

$$|p(z + i\eta) - p(z)| < \varepsilon \tag{4.8}$$

uniformly for all z such that $|\operatorname{Re}(z)| < \beta$.

<u>Theorem 4.3. If the cardinal function</u> p, <u>with related</u> α <u>as in</u> (1.4), <u>is almost periodic in some strip</u> $|\operatorname{Re}(z)| < \beta$ <u>with</u> $\beta > \alpha$, <u>then</u> E_p <u>is a Schauder basis for</u> $L^2[-\pi,\pi]$.

<u>Proof.</u> Let $\alpha < \rho < \beta$ and let C_0 be a simple path joining Γ_ρ to Γ_ρ which does not meet Z_p. Then for some $\varepsilon_0 > 0$

$$|p(z)| > \varepsilon_0, \quad z \in C_0 .$$

Let $0 < \varepsilon < \varepsilon_0/2$ and let $\ell(\varepsilon,\beta)$ be selected as indicated above. Let $L > \ell + \delta$, where δ is a fixed non-negative number, and for each non-zero integer $k = \pm 1, \pm 2, \cdots$, let $\eta_k \in ((k-1)L + \delta, kL]$ be such that (4.8) holds with η replaced by η_k. Then let

$$C_k = \{z + i\eta_k \mid z \in C_0\}, \quad k = \pm 1, \pm 2, \cdots, \tag{4.9}$$

and it is clear that for all such k

$$|p(z)| > \varepsilon, \quad z \in C_k . \tag{4.10}$$

For each positive k let $\Gamma_{\rho,k}$ consist of the portion of Γ_ρ between C_{k-1} and C_k and for each integer pair k,ℓ, $k > \ell$, let $\Gamma_{\rho,k,\ell}$ be the portion of Γ_ρ between C_k and C_ℓ; thus $\Gamma_{\rho,k} = \Gamma_{\rho,k,k-1}$. Define also

$$R_{\rho,k} = \Gamma_{\rho,k} + C_k - C_{k-1}, \quad k = 1,2,3,\cdots,$$

$$R_{\rho,k,\ell} = \Gamma_{\rho,k,\ell} + C_k - C_\ell, \quad k,\ell = \pm 1,\pm 2,\cdots, \quad k > \ell.$$

For z outside $R_{\rho,k,\ell}$, which includes $|\mathrm{Re}(z)| > \rho$, define

$$\psi_{k,\ell}(z) = \frac{1}{2\pi i} \int_{R_{\rho,k,\ell}} \frac{\psi(\zeta)}{\zeta - z}\, d\zeta.$$

Extended by analytic continuation to $C - (Z_p \cap \mathrm{Int}\, R_{\rho,k,\ell})$, $\psi_{k,\ell} \in R_p$.
Similarly define $\psi_{k,\ell} \in \Psi$, but not necessarily to Ψ_p, by

$$\tilde{\psi}_{k,\ell}(z) = \frac{1}{2\pi i} \int_{R_{\rho,k,\ell}} \frac{\psi(\zeta)}{\zeta - z}\, d\zeta.$$

Let $\rho < \sigma < \beta$. It is an easy consequence of the properties of the H^2 spaces in a half plane (see e.g., [B]) or the Carleson measure theorem ([L], [M]) that

$$\lim_{\substack{k\to\infty \\ \ell\to-\infty}} \left| \psi - \tilde{\psi}_{k,\ell} \right|_\sigma = 0. \tag{4.11}$$

Since $\psi(z) = \phi(z)/p(z)$, $\phi \in \Phi$, the Riemann-Lebesgue lemma shows that

$$\lim_{|k| \to \infty} \left(\sup_{\zeta \in C_k} |\psi(\zeta)| \right) \equiv \sup_{|k| \to \infty} c_k = 0$$

Since

$$\left| \psi_{k,\ell}(z) + \tilde{\psi}_{k,\ell}(z) \right| \leq \frac{(c_k + c_\ell)\ell(C)}{2\pi d(z, \Gamma_{\rho,k,\ell})}$$

where $\ell(C)$ is the length of C and ($I \equiv$ distance)

$$d(z, \Gamma_{\rho,k,\ell}) = \min_{\zeta \in \Gamma_{\rho,k,\ell}} |\zeta - z|$$

it is clear that

$$\lim_{\substack{k\to\infty \\ k\to-\infty}} \| \psi_{k,\ell} \quad \tilde{\psi}_{k,\ell} \|_\sigma = 0$$

and therefore, from (4.11)

$$\lim_{\substack{k\to\infty \\ \ell\to-\infty}} \|\psi - \psi_{k,\ell}\|_\sigma = 0 \,.$$

If we let

$$\psi = \mathcal{L} g, \quad \psi_{k,\ell} = \mathcal{L} g_{k,\ell}$$

then $\psi_{k,\ell} \in R_p \Rightarrow g_{k,\ell} \in E_p$. Since $\|\psi - \psi_{k,\ell}\|_\sigma$ is equivalent to $\|g - g_{k,\ell}\|_{L^2[-\pi,\pi]}$ and since

$$g_{k,\ell} = \sum_{j=\ell+1}^{k} g_j \,, \quad \hat{g}_j = {}^{-1}\, \hat{\psi}_j$$

$$\hat{\psi}_j = \frac{1}{2\pi i} \int_{\Gamma_{\rho,j}} \frac{\psi(\zeta)}{\zeta - z}\, d\zeta \in R_p \,.$$

we have

$$g = \sum_{j=-\infty}^{\infty} \hat{g}_j \,, \quad \hat{g}_j \in E_p \,.$$

convergent in $L^2[-\pi,\pi]$, and the proof is complete.

We will have more to say about the significance of the assumption about the almost periodicity of p in the concluding remarks of Section 5.

Series in the functions $e_{k,\nu}$ described by (4.7) have been referred to in the literature as <u>nonharmonic Fourier series</u>. Much of the interest in such series centers on the question of whether or not they form a <u>Riesz basis</u> for $L^2[-\pi,\pi]$. A sequence of elements, $\{x_k\}$, in a Hilbert space X forms a Riesz basis for X if it is a Schauder basis for X and, with

$$x = \sum_{k\in K} c_k x_k \tag{4.12}$$

the unique series representation of x in terms of this basis, there are positive numbers b, B, independent of x, such that

$$b^{-2} |x|^2 \le \sum_{k\in K} |c_k| \le B^2 |x|^2 \,. \tag{4.13}$$

It is evident that $\{x_k\}$ is a Riesz basis if and only if the map from $\{c_k\} \in \ell_K$ to $x \in X$ defined by (4.12) is bounded and boundedly invertible.

A generalization of the Riesz basis notion is that of a uniform decomposition of X. Suppose X_k, $k \in K$, is a sequence of subspaces of X. If every $x \in X$ can be written uniquely as an X-convergent series

$$x = \sum_{k \in K} \xi_k, \qquad \xi_k \in X_k, \tag{4.14}$$

and, with b, B positive and independent of x

$$b^{-2} | x |^2 < \sum_{k \in K} | \xi_k |^2 < B^2 | x |^2 ,$$

then we will say that the X_k form a uniform decomposition of X. A special case occurs when $\{x_k\}$ is a Riesz basis for X and $X_k = [x_k]$ so that for each k

$$\xi_k = c_k x_k$$

for some complex scalars c_k, $k \in K$.

It is well known that if $\{x_k\} \subset X$ is a strongly independent Schauder basis for X, and if Ξ is a representation of X' relative to the bilinear form $\langle x, \xi \rangle$, then there are unique $\xi_k \in \Xi$ such that

$$\langle x_k, \xi_\ell \rangle = \begin{cases} 1, & k = \ell \\ 0, & k \neq \ell \end{cases} \qquad k, \ell \in K.$$

When $\{x_k\}$ is a Riesz basis for X, $\{\xi_k\}$ is a Riesz basis for Ξ. The comparable notions for a uniform decomposition are as follows. For each k we have

$$X = X_k \oplus \hat{X}_k$$

where $\hat{X}_k$ is the closed span of the X_ℓ, $\ell \neq k$. Thus there is a unique decomposition

$$x = x_k + \hat{x}_k , \quad x_k \in X_k, \quad \hat{x}_k \in \hat{X}_k .$$

Let

$$P_k x = x_k .$$

Then P_k is a bounded projection with range X_k and $I-P_k$ is a bounded projection with range X_k. We define Ξ_k to be the range of the dual projection P_k on Ξ and we define Ξ_k to be the range of $I-P_k$ in Ξ. Clearly for $x \in X_k$, $\xi \in \Xi_k$ we have

$$\langle x,\xi\rangle = \langle P_k x,(I-P_k')\xi\rangle = \langle (P_k-P_k^2)x,\xi\rangle = \langle 0,\xi\rangle = 0.$$

and we have a similar relation for $x \in X_k$, $\xi \in \Xi_k$.

If for every $x \in X$ we are assured of the existence of a unique, convergent representation (4.14), whether (4.15) holds or not, we will say that the X_k form a Schauder decomposition of X.

Let us now place Theorem 4.3 in the context which we have just developed. For each integer k we define a linear operator, P_k, on Ψ_p, by

$$(P_k\psi)(z) = \frac{1}{2\pi i} \int_{R_{\rho,k}} \frac{\psi(\zeta)d\zeta}{z-\zeta} , \quad z \in \mathrm{Ext}(R_{\rho,k}).$$

Setting

$$\psi_k(z) = (P_k\psi)(z), \qquad k = 0,\pm1,\pm2,\cdots$$

$$\Psi_{p,k} = P_k\Psi_p , \qquad k = 0,\pm1,\pm2,\cdots \tag{4.17}$$

$\Psi_{p,k}$ consists of rational functions $\rho(z) = \sigma(z)/\tau(z)$, where $\sigma(z)$ and $\tau(z)$ are polynomials with $\deg \sigma < \deg \tau$. Moreover, $p(z)(\sigma(z)/\tau(z))$ is entire. The dual operators defined on Φ will be called P_k'. Their definition is

$$\phi_k(z) = (P_k'\phi)(z) = p(z)\,\frac{1}{2\pi i} \int_{R_{\rho,k}} \frac{\phi(\zeta)d\zeta}{(z-\zeta)p(\zeta)} , \qquad z \in \mathrm{Ext}(R'_{\rho,k}). \qquad k = 0,\pm1,\pm2,\cdots \tag{4.18}$$

and we define

$$\Phi_{p,k} = P_k'\Phi, \quad k = 0, \pm 1, \pm 2 \cdots \tag{4.19}$$

The proof that ϕ_k, as defined here, has an entire analytic continuation lying in Φ, so that $\Phi_{p,k}$ is a subspace of Φ, follows much the same lines as Theorem 2.3

It is easy to see that the operators P_k, P_k' do not depend on the particular choice of $\rho > \alpha$.

Proposition 4.4. The operators P_k, P_k' are projections on Ψ_p, Φ, respectively, and for

$$\psi \in \Psi_{p,k}, \quad \phi \in \Phi_\ell, \quad \ell \neq k,$$

we have $\langle \phi, \psi \rangle = 0$. Moreover, P_k is the dual operator to P_k in the sense that for $\phi \in \Phi$, $\psi \in \Psi$,

$$\langle P_k'\phi, \psi \rangle = \langle \phi, P_k \psi \rangle .$$

The proofs are easy and essentially the same as those given in connection with the operator calculus in [N] and are omitted.

Theorem 4.5. Under the hypotheses of Theorem 4.3, taking δ, described preceding (4.9), so that $\delta > 1$, the spaces $\Psi_{p,k}$ described by (4.16), (4.17) form a uniform decomposition of the Hilbert space Ψ_p.

Proof. We need a standard parametrization of the paths $R_{\rho,k}$. Let the points where C_k meets Γ_ρ and Γ_ρ, respectively, be $\overset{+}{r}_{\rho,k}$, $\overset{-}{r}_{\rho,k}$. We construct a map

$$\zeta_k = \zeta_k(\zeta)$$

from $R_{\rho,0}$ onto $R_{\rho,k}$ as follows:

$$Z_k(\zeta) = \begin{cases} \zeta + i\eta_k, & \zeta \in C_0 \\ \zeta + (\eta_{k-1} - \eta_{-1}), & \zeta \in C_{-1} \,. \end{cases}$$

The vertical sides of $R_{\rho,k}$ are $\Gamma^+_{\rho,k}$, $\Gamma^-_{\rho,k}$. We define

$$\zeta_k(\zeta) = \begin{cases} r^+_{\rho,k-1} + (\zeta - r^+_{\rho,-1})\left(\dfrac{r^+_{\rho,k} - r^+_{\rho,k-1}}{r^+_{\rho,0} - r^+_{\rho,-1}}\right), & \zeta \in \Gamma^+_{\rho,0} \\ r^-_{\rho,k-1} + (\zeta - r^-_{\rho,-1})\left(\dfrac{r^-_{\rho,k} - r^-_{\rho,k-1}}{r^-_{\rho,0} - r^-_{\rho,-1}}\right), & \zeta \in \Gamma^-_{\rho,0} \end{cases}$$

The construction of the paths C_k is such that the lengths of $\Gamma_{\rho,k}$, $\Gamma_{\rho,k}$, i.e., $|r_{\rho,k} - r_{\rho,k-1}|$ and $|r_{\rho,k} - r_{\rho,k-1}|$, always lie in the interval $[\delta, 2L-\delta]$. It thus follows that $|\zeta_k(\zeta)|$ is bounded and bounded away from zero, uniformly with respect to k and $\zeta \in C_0$. Write

$$d < |\zeta_k'(\zeta)| < D. \tag{4.20}$$

Let ψ be an element of Ψ_p. Then

$$\psi_k(z) = (P_k\psi)(z) = \frac{1}{2\pi i}\int_{R_{\rho,k}} \frac{\psi(\omega)d\omega}{z-\omega}\,, \qquad z \in \mathrm{Ext}(R_{\rho,k}). \tag{4.21}$$

Setting

$$\omega = \zeta_k(\zeta), \qquad d\omega = \zeta_k'(\zeta)\, d\zeta$$

we can re-express (4.20) as (suppressing the argument ζ)

$$\psi_k(z) = \frac{1}{2\pi i}\int_{R_{\rho,0}} \frac{\psi(\zeta_k)\zeta_k'\, d\zeta}{z - \zeta_k}\,, \quad z \in \mathrm{Ext}(R_{\rho,k}). \tag{4.22}$$

For $\sigma > \rho$ we will estimate

$$\sum_{k=-\infty}^{\infty} \|\psi_k\|_\sigma^2 = \sum_{k=-\infty}^{\infty} \int_{\Gamma_\sigma} |\psi_k(z)|^2\, dz.$$

Let us note that for fixed $\zeta \in R_{\rho,0}$, $\zeta_k(\zeta)$, $\zeta_{k-1}(\zeta)$ have the same real part and

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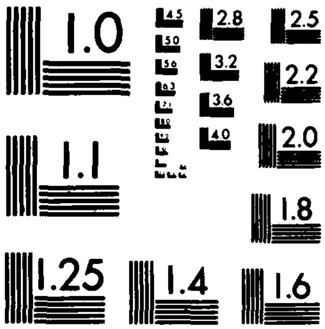

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$$\operatorname{Im}(\zeta_k(\zeta) - \zeta_{k-1}(\zeta)) \geq \delta > 1.$$

Then we estimate

$$\sum_{k=-\infty}^{\infty} \int_{\Gamma_\sigma} |\psi_k(z)|^2 |dz| = \frac{1}{4\pi^2} \sum_{k=-\infty}^{\infty} \int_{\Gamma_\sigma} \left| \int_{R_{\rho,0}} \frac{\psi(\zeta_k)\zeta_k'}{z-\zeta_k} d\zeta \right|^2 dz$$

$$\leq \frac{D^2}{4\pi^2} \sum_{k=-\infty}^{\infty} \int_{\Gamma_\sigma} \left(\int_{R_{\rho,0}} \left| \frac{\psi(\zeta_k)}{z-\zeta_k} \right|^2 |d\zeta| \int_{R_{\rho,0}} |d\zeta| \right) |dz| \leq$$

$$\frac{D^2 \ell(R_{\rho,0})}{4\pi^2} \sum_{k=-\infty}^{\infty} \int_{R_{\rho,0}} \int_{\Gamma_\sigma} \left| \frac{\psi(\zeta_k)}{z-\zeta_k} \right|^2 |dz| \quad d|\zeta|$$

$$= \frac{D^2 \ell(R_{\rho,0})^2}{2\pi^2} \sup_{\zeta \in R_{\rho,0}} \left\{ \sum_{k=-\infty}^{\infty} \int_{\Gamma_\sigma} \left| \frac{\psi(\zeta_k)}{z-\zeta_k} \right|^2 |dz| \right\}$$

$$\leq \frac{D^2 \ell(R_{\rho,0})^2}{2\pi^2 (\inf_{z \in R_{\rho,k}} |p(z)|)^2} \int_{-\infty}^{\infty} \frac{dr}{(\sigma-\alpha)^2 + r^2} \sup_{\zeta \in R_{\rho,0}} \left\{ \sum_{k=-\infty}^{\infty} |\phi(\zeta_k)|^2 \right\}, \quad (4.24)$$

where

$$\psi(z) = \phi(z)/p(z), \quad \phi \in \Phi .$$

Clearly our task is to estimate the sum in (4.24). Let

$$\tilde{\psi}(w) = \sum_{k=-\infty}^{\infty} \frac{\overline{\phi(\zeta_k)}}{\zeta_k - w} .$$

From (4.23) and the inequalities (3.28) there are positive numbers $\tilde{c}, \tilde{C}$ such that

$$\tilde{c}^{-2} \|\tilde{\psi}\|_\sigma^2 \leq \sum_{k=-\infty}^{\infty} |\phi(\zeta_k)|^2 \leq \tilde{C}^2 \|\tilde{\psi}\|_\sigma^2 .$$

Then it is easy to see that

$$\sum_{k=-\infty}^{\infty} |\phi(\zeta_k)|^2 = \frac{1}{2\pi i} \int_{\Gamma_\sigma} \phi(w) \quad \tilde{\psi}(w) dw$$

$$\leq \frac{1}{2\pi} \|\phi\|_\sigma \quad \|\psi\|_\sigma \leq \frac{\tilde{C}}{2\pi} \|\phi\|_\sigma \left(\sum_{k=-\infty}^{\infty} |\phi(\zeta_k)|^2 \right)^{1/2}$$

and we conclude that

$$\sum_{k=-\infty}^{\infty} \|\phi(\zeta_k)\|^2 < \frac{\tilde{c}^2}{4\pi^2} |\phi|_\sigma^2 \tag{4.25}$$

the constant c depending only on $\delta > 1$, and not on the particular sequence ζ_k. Then from (4.24), (4.25) we have,

$$\sum_{k=-\infty}^{\infty} \|\psi_k\|_\sigma^2 = \sum_{k=-\infty}^{\infty} \int_{\Gamma_\sigma} |\psi_k(z)|^2 |dz|$$

$$< \left(\frac{D^2 \ell(R_{\rho,0})^2 \tilde{c}^2}{8\pi^4} \int_{-\infty}^{\infty} \frac{dr}{(\sigma-\alpha)^2+r^2} \right) \left(\frac{\sup_{z\in\Gamma_\sigma} |p(z)|}{\inf_{z\in R_{\rho,k}} |p(z)|} \right)^2 |\psi|_\sigma^2$$

$$\equiv C_1^2 \|\psi\|_\sigma^2 . \tag{4.26}$$

To obtain an inequality in the other direction we note that

$$\|\psi\|_\sigma^2 = \int_{\Gamma_\sigma} |\psi(z)|^2 |dz| = \int_{\Gamma_\sigma} \left| \sum_{k=-\infty}^{\infty} \psi_k(z) \right|^2 |dz|$$

$$= \int_{\Gamma_\sigma} \left| \sum_{k=-\infty}^{\infty} \frac{1}{2\pi i} \int_{R_{\rho,0}} \frac{\psi_k(\zeta_k)\zeta_k' d\zeta}{z-\zeta_k} \right|^2 |dz|$$

$$< \frac{1}{4\pi^2} \int_{\Gamma_\sigma} \sum_{k=-\infty}^{\infty} \sum_{\ell=-\infty}^{\infty} \int_{R_{\rho,0}} \int_{R_{\rho,0}} \frac{\psi(\zeta_k)\overline{\psi_\ell(\zeta_\ell)}\zeta_k'\overline{\zeta_\ell'} d\zeta\, \overline{d\tilde{\zeta}}}{(z-\zeta_k)\overline{(z-\zeta_\ell)}} |dz|$$

$$= \frac{1}{4\pi^2} \int_{R_{\rho,0}} \int_{R_{\rho,0}} \int_{\Gamma_\sigma} \sum_{k=-\infty}^{\infty} \sum_{\ell=-\infty}^{\infty} \frac{\psi_k(\zeta_k)\overline{\psi_\ell(\zeta_\ell)}\zeta_k'\overline{\zeta_\ell'} |dz|}{(z-\zeta_k)\overline{(z-\zeta_\ell)}} d\zeta\, \overline{d\tilde{\zeta}}$$

$$= \frac{1}{4\pi^2} \int_{R_{\rho,0}} \int_{R_{\rho,0}} \int_{\Gamma_\sigma} \left(\sum_{k=-\infty}^{\infty} \frac{\psi_k(\zeta_k)\zeta_k'}{(z-\zeta_k)} \right) \left(\sum_{\ell=-\infty}^{\infty} \frac{\overline{\psi_\ell(\zeta_\ell)\zeta_\ell'}}{\overline{(z-\zeta_\ell)}} \right) |dz|\, d\zeta\, \overline{d\tilde{\zeta}}$$

$$< \frac{D^2 \ell(R_{\rho,0})^2}{4\pi^2} \left(\sup_{\zeta\in R_{\rho,0}} \left| \sum_{k=-\infty}^{\infty} \frac{\psi_k(\zeta_k)}{z-\zeta_k} \right|_\sigma^2 \right).$$

Since

$$\psi_k(\zeta_k) = \frac{1}{p(\zeta_k)} \frac{1}{2\pi i} \int_{\Gamma_\sigma} \frac{p(\zeta)\psi_k(\zeta)}{\zeta-\zeta_k} d\zeta$$

we have

$$|\psi_k(\zeta_k)| \le \frac{1}{\sqrt{2}\,\pi}\left(\int_{-\infty}^{\infty}\frac{dr}{(\sigma-\alpha)^2+r^2}\right)^{1/2}\left(\frac{\sup_{z\varepsilon\Gamma_\sigma}|p(z)|}{\inf_{z\varepsilon R_{\rho,0}}|p(z)|}\right)\|\psi_k\|_\sigma$$

and then

$$\sum_{k=-\infty}^{\infty}|\psi_k(\zeta_k)|^2 \le \frac{1}{2\pi^2}\int_{-\infty}^{\infty}\frac{dr}{(\sigma-\alpha)^2+r^2}\left(\frac{\sup_{z\varepsilon\Gamma_\sigma}|p(z)|}{\inf_{z\varepsilon R_{\rho,k}}|p(z)|}\right)\sum_{k=-\infty}^{\infty}\|\psi_k\|_\sigma^2 .$$

But we know from (3.28) that

$$\left\|\sum_{k=-\infty}^{\infty}\frac{\psi_k(\zeta_k)}{z-\zeta_k}\right\|_\sigma^2 \le \tilde{C}^2\sum_{k=-\infty}^{\infty}|\psi_k(\zeta_k)|^2$$

with c depending only on $\delta > 1$, and hence

$$\|\psi\|_\sigma^2 \le \frac{D^2\ell(R_{\rho,0})^2}{4\pi^2}\left(\sup_{\zeta\varepsilon R_{\rho,0}}\left\|\sum_{k=-\infty}^{\infty}\frac{\psi_k(\zeta_k)}{z-\zeta_k}\right\|_\sigma^2\right)$$

$$\le \frac{D^2\ell(R_{\rho,0})^2\ \tilde{c}^2}{4\pi^2}\sum_{k=-\infty}^{\infty}|\psi_k(\zeta_k)|^2$$

$$\le \frac{D^2\ell(R_{\rho,0})^2\ \tilde{c}^2}{8\pi^4}\left(\int_{-\infty}^{\infty}\frac{dr}{(\sigma-\alpha)^2+r^2}\right)\left(\frac{\sup_{z\varepsilon\Gamma_\sigma}|p(z)|}{\inf_{z\varepsilon R_{\rho,k}}|p(z)|}\right)\sum_{k=-\infty}^{\infty}\|\psi_k\|_\sigma^2$$

$$= C_1^2\sum_{k=-\infty}^{\infty}|\psi_k|_\sigma^2 , \tag{4.28}$$

which completes the proof.

We now address ourselves to the question as to when the individual functions (4.4) form a Riesz basis for $L^2[-\pi,\pi]$. The next theorem treats the case wherein the cardinal function p is almost periodic and the zeros, z_k, of p are simple.

Theorem 4.6. Let the cardinal function p *satisfy the hypotheses of Theorem 4.3 and let the zeros,* z_k, $k \varepsilon K$, *of* p *be simple. Suppose there are positive numbers* r, R *such that*

$$r < \left| p'(z_k) \right| < R, \quad k \varepsilon K. \tag{4.29}$$

Then the functions

$$\psi_k(z) = \frac{1}{z - z_k} \tag{4.30}$$

form a Riesz basis for Ψ_p.

Remark. The right hand inequality in (4.29) follows, of course, from the boundedness of p in strips $\left| Re(z) \right| < \rho$.

Proof of Theorem 4.6. From Theorem 4.3 the functions $e^{z_k t}$ form a Schauder basis for $L^2[-\pi, \pi]$; equivalently, the functions (4.30) form a Schauder basis for Ψ_p.

Now consider sequences of coefficients $\{a_k \mid k \varepsilon K\} \varepsilon \ell_K$, and define the operator $T : \ell_K \rightarrow \Psi_p$ by

$$T(\{a_k\}) = \sum_{k \varepsilon K} \frac{a_k}{z - z_k} \equiv \sum_{k \varepsilon K} a_k \psi_k(z). \tag{4.31}$$

The domain of T consists of all $\{a_k\}$ for which the right hand side is convergent in Ψ_p. Thus T is densely defined (look at finite sequences), one to one (by strong independence), and has dense range (by completeness). The adjoint map is

$$T^* : \phi \varepsilon \Phi \rightarrow \{\phi(z_k) \mid k \varepsilon K\}. \tag{4.32}$$

Since

$$\phi_k(z) = \frac{p(z)}{p'(z_k)(z-z_k)}$$

is the unique element of Φ biorthogonal to ψ_k, i.e.

$$\langle \phi_k, \psi_\ell \rangle = \delta_{k\ell} = \begin{cases} 1, & k = \ell , \\ 0, & k \neq \ell , \end{cases}$$

and $\phi_k(z_\ell) = \delta_{k\ell}$, it is easy to see that T^* is defined on sums $\sum b_k \phi_k(z)$ and hence has dense range. That it is one to one follows from the proof of Theorem 2.5.

Now, in fact, T and T^* are both bounded. The boundedness of T^* follows from the fact that, for $\phi \in \Phi$, $e^{-\pi z}\phi(z)$ lies in the Hardy space $H^2\{\mathrm{Re}(z) > \rho\}$ and the fact that the zeros z_k of an entire function $\phi \in \Phi$ have a maximum density; given $L > 0$, there is an $M > 0$ such that the number of zeros z_k in any rectangle $|\mathrm{Im}(z-a)| < \ell$, $|\mathrm{Re}(z)| < \rho$ does not exceed $M\ell$ when $\ell > L$. The Borel measure on $\mathrm{Re}(z) > -\rho$ defined by

$$\mu(z_k) = 1, \quad z_k \in Z_p ,$$

$$\mu(\{\mathrm{Re}(z) > -\rho\} - Z_p) = 0,$$

is then a Carleson measure ([L], [M]) and there is a $B > 0$ such that for $\phi \in \Phi$

$$\sum_{k=-\infty}^{\infty} |\phi(z_k)|^2 \leq B \|\phi\|_\rho^2 ,$$

i.e., T^* has range entirely included in ℓ_K^2 and is a bounded linear transformation. But then $T = (T^*)^*$ is a bounded linear transformation.

To show the Riesz basis property it is only necessary to establish that T^{-1} is bounded. We have seen from the boundedness of T that if $\{a_k\} \in \ell_K^2$ then

$$\psi(z) = \sum_{k\epsilon K} \frac{a_k}{z-z_k} \quad \epsilon \ \Psi_p$$

and we have

$$| \psi |_\rho < | T | \sum_{k\epsilon K} | a_k |^2 .$$

Now let $\{b_k\} \ \epsilon \ \ell_K$. Then

$$(T^*)^{-1}\{b_k\} = \sum_{k\epsilon K} b_k \phi_k(z) = \sum_{k\epsilon K} b_k \frac{p(z)}{p'(z_k)(z-z_k)} ,$$

the domain of $(T^*)^{-1}$ being those $\{b_k\} \ \epsilon \ \ell^2_K$ for which the series on the right converges. But (cf. (2.20) for definition of P)

$$\sum_{k\epsilon K} b_k \frac{p(z)}{p'(z_k)(z-z_k)} = p(z) \sum_{k\epsilon K} \left(\frac{b_k}{p'(z_k)}\right) \frac{1}{z-z_k}$$

$$= P\left(\sum_{k\epsilon K} \left(\frac{b_k}{p'(z_k)}\right) \frac{1}{z-z_k}\right) = PT\left\{\frac{b_k}{p'(z_k)}\right\}.$$

Since the numbers $p'(z_k)$ are bounded away from zero, the map

$$C\{b_k\} = \left\{\frac{b_k}{p'(z_k)}\right\}$$

is bounded on ℓ_K. Thus

$$(T^*)^{-1}\{b_k\} = PT\left\{\frac{b_k}{p'(z_k)}\right\} = PTC\{b_k\},$$

i.e.,

$$(T^*)^{-1} = PTC .$$

Since P, T and C are all bounded, we conclude that $(T^*)^{-1}$, and hence T^{-1}, is bounded. Hence $\{(z-z_k)^{-1}\}$ is the image of the standard orthonormal basis for ℓ^2 under the bounded and boundedly invertible linear transformation T and we conclude that $\{(z-z_k)^{-1}\}$ is a Riesz basis for Ψ_p.

Corollary 4.7. Under the hypotheses of Theorem 4.6 *the exponentials* $\{e^{z_k t} \mid k \in K\}$ *form a Riesz basis for* $L^2[-\pi,\pi]$.

This is an immediate consequence of the fact that for $g \in L_\rho(-\infty,\infty)$ such that $(g) = \psi \in \Psi_p$, the norms $\|g\|_{L^2[-\pi,\pi]}$ and $\|\psi\|_\rho$, or $\|\psi\|_\Phi$, are equivalent, together with $(z-z_k)^{-1} = (e^{z_k t})$.

5. Concluding Remarks.

If we agree to refer to the Schauder bases of exponentials $\{e^{z_k t}\}$ for $L^2[-\pi,\pi]$ associated with the zeros of a regular cardinal function p, as defined in Section 1, as generating regular nonharmonic Fourier series, we obtain a class of such series which overlaps, but is neither included in, nor includes, the class of such series studied in the familiar literature on the subject. In the classical literature, which includes, e.g. [C], [D], [E], [F], [F], [J], [O], and numerous other contributions, the emphasis lies on properties of the sequence $\{z_k\}$; properties such as density, asymptotic gap, proximity to the imaginary integers ik, etc., are the starting point. What we call the cardinal function, p, is constructed as an infinite product

$$p(z) = \prod_{k\in K} \left(1 - \frac{z}{z_k}\right),$$

ordinarily with grouping of terms to ensure convergence. The properties of p are then deduced from the properties of the sequence $\{z_k\}$.

The most frequently studied sequences $\{z_k\}$ (see, e.g. [E], [O]) are those imaginary sequences for which (letting K = the integers now)

$$\sup_{-\infty < k < \infty} \left| z_k - ki \right| = \gamma < \frac{1}{4} . \tag{5.1}$$

Not all of these nonharmonic Fourier series are encompassed in our framework. The property of prime importance for p, referring to our framework now, is that p itself should not lie in $L^2(\Gamma_\rho)$ but, for each zero z_k of p,

$$\phi_k(z) = \frac{p(z)}{p'(z_k)(z-z_k)}$$

should lie in that space. This requirement, by itself, does not make p bounded and bounded below on Γ_ρ as in our work here. Roughly speaking, it admits functions $p(z)$ whose growth on Γ_ρ is like $|z|^\mu$ with $-1/2 < \mu < 1/2$. Such growth is obtained for sequences (5.1), e.g., if

$$z_k \sim ik\left(1 - \frac{\mu}{2|k|}\right), \qquad |k| \to \infty.$$

Consequently, such cases are not covered by our theory as presented in this paper; we hope to be able to modify our methods to cover them.

To give an idea of what our theory does encompass, we first need a reasonably large class of cardinal functions which meet our conditions. Such a class may be constructed as follows. Consider the distribution, d, with support in $[-\pi,\pi]$, defined by

$$d = \delta_{(\pi)} + c_0\delta_{(-\pi)} + \sum_{k=1}^{\infty} c_k\, \delta_{(\xi_k)} + f, \tag{5.2}$$

where $\delta_{(\xi)}$ is the Dirac distribution with support ξ, $c_0 \neq 0$,

$$\sum_{k=1}^{\infty} |c_k| < \infty .$$

the points ξ_k are distinct points in $(-\pi,\pi)$, and $f \in L^1[-\pi,\pi]$. Using results from [N] it may be shown that the Fourier transform of this distribution

$$p(z) = \langle d, e^z \rangle, \tag{5.3}$$

is almost periodic in any strip $|\mathrm{Re}(z)| \leq \rho$, $\rho > 0$, in the complex plane. It is also easy to see that the conditions (1.3), (1.4) are met for some $\alpha > 0$. Thus $p(z)$ as defined by (5.3) is a (regular) cardinal function as defined in this paper.

A very interesting case, not covered in the classical treatments [C], [D], and [E], but presented as an unproved theorem in [F], occurs when the series in (5.2) is finite, say of length $N-1$, and

$$\xi_k = -\pi + \frac{k}{N} 2\pi , \quad k = 1,2,\cdots,N-1. \tag{5.4}$$

In this case

$$p(z) = e^{\pi z} + c_0 e^{-\pi z} + \sum_{k=1}^{N-1} c_k e^{\xi_k z} + \int_{-\pi}^{\pi} e^{zt} f(t)dt$$

$$\equiv p_0(z) + \int_{\pi}^{\pi} e^{zt} f(t)dt. \tag{5.5}$$

The zeros of $p_0(z)$ then take the form $z_{j\ell} = \log(\zeta_j) + 2\pi\ell i$, $j = 1,2,\cdots,N$, $-\infty < \ell < \infty$, where the ζ_j are the zeros of the polynomial

$$\zeta^N + c_{N-1}\zeta^{N-1} + \cdots + c_1\zeta + c_0$$

and the principal value of the logarithm is intended. The zeros, $z_{j\ell}$, of $p(z)$ are easily shown to be asymptotic to the $z_{j\ell}$ as $|z_{j\ell}| \to \infty$. Theorem 4.6 applies here if the $z_{j\ell}$ are all simple zeros.

An important case also arises for $p(z)$ having the form (5.5) but with the ξ_k not rationally related to π, so that, in particular, (5.4) does not obtain. In this case we cannot give a simple asymptotic expression for the zeros of $p(z)$ and they may cluster in various complicated ways as $|z| \to \infty$. Nevertheless, $p(z)$ remains almost periodic in strips $|\mathrm{Re}(z)| < \rho$, $\rho > 0$, and Theorem 4.5 applies to show that $L^2[-\pi,\pi]$ admits a uniform decomposition in terms of finite dimensional subspaces spanned by generalized exponentials associated with the zeros of p. This result has a number of uses in connection with the theory of linear symmetric hyperbolic systems of partial differential equations having wave speeds which are not rationally related (see, e.g. [P]).

It is clear, when $p(z)$ has the form (5.3), that the associated generalized exponentials are the exponential solutions of the scalar neutral functional equation

$$w(t+\pi) + c_0 w(t-\pi) + \sum_{k=1}^{\infty} c_k w(t+\xi_k) + \int_{-\pi}^{\pi} f(s)w(t+s)ds = 0. \tag{5.6}$$

As such, these generalized exponentials, restricted to $[-\pi,\pi]$, are the generalized eigenfunctions of the operator

$$(Aw)(x) = w'(x) \tag{5.7}$$

with (A) consisting of those functions w in the Sobolev space $H^1[-\pi,\pi]$ which satisfy the boundary condition

$$w(\pi) + c_0 w(-\pi) + \sum_{k=1}^{\infty} c_k w(\xi_k) + \int_{-\pi}^{\pi} f(s)w(s)ds = 0. \tag{5.8}$$

It is well known that when $c_0 \neq 0$, which we assume, the operator (5.7) generates a strongly continuous group of bounded operators on $L^2[-\pi,\pi]$. This group has been studied in [Q], where it has also been shown that there is a very strong connection between any exponential Riesz basis for $L^2[-\pi,\pi]$ and a corresponding group of restricted shifts, or translations. This is another topic which we hope to return to at another time.

In this connection it is, of course clear that our methods are quite similar to the methods used for studying the spectral properties of differential operators which involve various contour integration methods applied to the resolvent operator $(zI-A)^{-1}$ (see [R], e.g.). The meromorphic function $1/p(z)$ plays much the same role as the resolvent does in that theory. In fact it is shown in [Q] that for $p(z)$ having the form (5.3), and A the operator (5.7) with domain characterized by the boundary condition (5.8), that for $\psi = (w) \in \Psi_p$, we have

$$\psi(z) = ((zI-A)^{-1}w)(0).$$

If one forms the distributional solution $w(t)$ of

$$w(t+\pi) + c_0 w(t-\pi) + \sum_{k=1}^{\infty} c_k w(t+\xi_k) + \int_{-\pi}^{\pi} f(s)w(t+s)ds = \delta_{(0)}$$

it may be seen that $1/p(z)$ is the Laplace transform of w. This leads to the formula.

$$\frac{1}{p(z)} = ((zI - A)^{-1}\delta_{(0)})(0),$$

if $(zI-A)^{-1}$ is appropriately extended to $H^{-1}[-\pi,\pi]$, which includes the distribution $\delta_{(0)}$.

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